Exact solutions to the Weighted Region Problem*

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Abstract

In this paper, we consider the Weighted Region Problem. In the Weighted Region Problem, the length of a path is defined as the sum of the weights of the subpaths within each region, where the weight of a subpath is its Euclidean length multiplied by a weight $\alpha \geq 0$ depending on the region. We study a restricted version of the problem of determining shortest paths through a single weighted rectangular region. We prove that even this very restricted version of the problem is unsolvable within the Algebraic Computation Model over the Rational Numbers (ACM$\mathbb{Q}$). On the positive side, we provide the equations for the shortest paths that are computable within the ACM$\mathbb{Q}$. Additionally, we provide equations for the bisectors between regions of the Shortest Path Map for a source point on the boundary of (or inside) the rectangular region.

1 Introduction

The Weighted Region Problem (WRP) [15] is a well-known geometric problem that, despite having been studied extensively, is still far from being well understood. Consider a subdivision of the plane into (usually polygonal) regions. Each region R_i has a weight $\alpha_i \geq 0$, representing the cost per unit distance of traveling in that region. Thus, a straight-line segment σ, of Euclidean length $|\sigma|$, between two points in the same region has weighted length $\alpha_i \cdot |\sigma|$ when traversing the interior of R_i, or $\min\{\alpha_i, \alpha_j\} \cdot |\sigma|$ if it goes along the edge between R_i and R_j. Then, the weighted length of a path through a subdivision is the sum of the weighted lengths of its subpaths through each face or edge. The resulting metric is called the *Weighted Region Metric*. The WRP entails computing a shortest path $\pi(s,t)$ between two given points s and t under this metric. We denote the weighted length of $\pi(s,t)$ by $d(s,t)$. Figure 1 shows how the shape of a shortest path changes as the weight of one region varies.

Existing algorithms for the WRP—in its general formulation—are approximate. Since the seminal work by Mitchell and Papadimitriou [15], with the first $(1+\varepsilon)$-approximation, several algorithms have been proposed, with improvements on running times, but always keeping some dependency on the vertex coordinates sizes and weight ranges. These methods are usually based on the continuous Dijkstra's algorithm, subdividing triangle edges in parts for which crossing shortest paths have the same combinatorial structure (e.g., [15]), or on adding Steiner points (e.g., see [1, 2, 3, 5, 18]). However, rather recently it has been proved that computing an exact shortest path between two points using the Weighted Region Metric, even if there are only three different weights, is an unsolvable problem in the Algebraic Computation Model over the Rational Numbers (ACM$\mathbb{Q}$) [6]. In the ACM$\mathbb{Q}$ one can compute exactly any number that can be obtained from rational numbers by applying a finite number of operations from $+, -, \times, \div$, and $\sqrt[k]{\ }$, for any integer $k \geq 2$. This provides a theoretical explanation for the lack of exact algorithms for the WRP, and justifies the study of approximation methods.

This also raises the question of which are the special cases for which the WRP can be solved exactly. Two natural ways to restrict the problem are by limiting the possible weights and by restricting the shape of the regions. For example, computing a shortest path among polygonal or curved obstacles can be seen as a variant of the WRP with weights in the set $\{1, \infty\}$. Efficient algorithms exists for this problem, culminating with the recent algorithms by Wang [19] for polygonal obstacles, and by Hershberger et al. [11] for shortest paths among curved obstacles. The case for polygonal regions with weights in $\{0, 1, \infty\}$ can be solved in $O(n^2)$ time [9] by constructing a graph known as the *critical graph*, an extension of the visibility graph. Other variants that can be solved exactly correspond to regions shaped as regular k-gons with weight ≥ 2 (since they can be considered as obstacles), or regions with two weights $\{1, \alpha\}$ consisting of parallel strips [16]. In the latter case, the angle of incidence in each of the strips is the same, so they can be rearranged so that they are all together, and the angle of incidence can be computed exactly using Snell's law of refraction.

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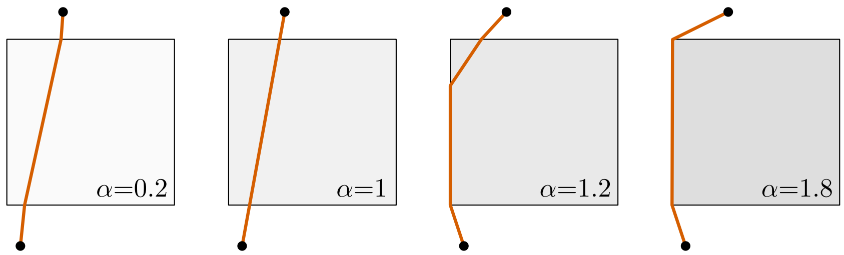

Figure 1: Examples of shortest paths between two points—shown in orange—for two weighted regions. The unbounded region has weight 1, the squares have varying weight α.

Our results. In light of the fact that the WRP is unsolvable in the ACM$\mathbb{Q}$ already for three different weights, in this work we study the case of two arbitrary weights, that is, weights in $\{1, \alpha\}$, where $\alpha \in \mathbb{Q}^+$. In particular, and without loss of generality, we assume that the weight of the unbounded region is 1. Otherwise, we could always rescale the weights to be 1 outside the regions. This case is particularly interesting, since an algorithm for weights $\{1, \alpha\}$ can be transformed into one for weights in $\{0, 1, \alpha, \infty\}$ [13]. However, the variant with weights $\{1, \alpha\}$ was conjectured to be as hard as the general WRP problem, see the first open problem in [9, Section 7]. (The results in [6] do not directly apply to weights $\{0, 1, \alpha, \infty\}$.)

This paper is organized as follows. First we present some preliminaries in Section 2. In Section 3 we consider two weights and one rectangular region R, with the source point s on its boundary or inside. For this setting, we figure out all types of possible optimal paths and give exact formulas to compute their lengths. In Section 3.3 we focus on the case where s is outside of R, and prove that in this case the WRP with weights $\{1, \alpha\}$ is already unsolvable in the ACM$\mathbb{Q}$, confirming the suspicions of Mitchell [13]. In Section 4 we explore the computation of the Shortest Path Map for s. We finish with some conclusions in Section 5.

2 Shortest paths and their properties

In this section we briefly review some key properties of shortest paths in weighted regions.

First, with our assumption that the weight within each region does not account for the effect of certain force fields that favors some directions of travel, shortest paths in the Weighted Region Problem will always be piecewise linear, see [15, Lemma 3.1]. Second, it is known that shortest paths must obey Snell's law of refraction. So we can think of a shortest path as a ray of light. Throughout this paper, the *angle of incidence* θ is defined as the minimum angle between the incoming ray and the vector perpendicular to the region boundary. The *angle of refraction* θ' is defined as the minimum angle between the outgoing ray and the vector perpendicular to the region boundary. Snell's law states that whenever the ray goes from one region R_i to another region R_j, then $\alpha_i \sin\theta = \alpha_j \sin\theta'$. In addition, whenever $\alpha_i > \alpha_j$, the angle θ_c at which $\frac{\alpha_i}{\alpha_j}\sin\theta_c = 1$ is called the *critical angle*. A ray that hits an edge at an angle of incidence greater than θ_c, will be totally reflected from the point at which it hits the boundary. In our problem, a shortest path will never be incident to an edge at an angle greater than θ_c.

Finally, if the space only contains orthoconvex regions[1] with weight at least $\sqrt{2}$, they can be simply considered as obstacles [16]. Thus, since we focus on a rectangular region R, we assume that its weight is $0 < \alpha < \sqrt{2}$. However, first we provide some general properties of shortest paths for arbitrary weighted regions that are interesting on their own.

Lemma 1 *Let $\mathcal{S}$ be a polygonal subdivision for which each region has a weight in the set $\{1, \alpha\}$, with $\alpha \geq 0$. A shortest path $\pi(s,t)$ visits any edge of the subdivision at most once.*

Proof. Assume, for the sake of contradiction, that $\pi(s,t)$ intersects an edge e in at least two disjoint intervals I_1 and I_3 (note that I_1 and I_3 could be points). Moreover, let $p_1 \in I_1$ and $p_3 \in I_3$ be points for which the subpath $\pi(p_1,p_3) \subseteq \pi(s,t)$ does not intersect e in any points other than p_1 and p_3. Let p_2 be a point on $\pi(p_1,p_3)$ between p_1 and p_3, which thus does not lie on e. Now observe that there exists a path $\overline{p_1p_3}$ from p_1 to p_3 of length $\min\{1,\alpha\}|\overline{p_1p_3}|$. Since p_2 does not lie on $\overline{p_1p_3}$, it follows by the triangle inequality that the length of $\pi(p_1,p_3)$ is strictly larger than $\min\{1,\alpha\}|\overline{p_1p_3}|$. Hence, $\pi(s,t)$ is not a shortest path, and we obtain a contradiction. □

Observe that the previous result is not true when there are more than two weights, see [15, Figure 2].

Corollary 2 *Let $\mathcal{S}$ be a polygonal subdivision with n vertices for which each region has a weight in the set $\{1, \alpha\}$, with $\alpha \geq 0$. Any shortest path $\pi(s,t)$ is a polygonal chain with at most $O(n)$ vertices.*

[1] A region is orthoconvex if its intersection with every horizontal and vertical line is connected or empty [17].

Proof. Any shortest path is a polygonal chain whose interior vertices all lie on edges of $\mathcal{S}$, see [15, Proposition 3.8]. By Lemma 1, each edge contributes with at most two vertices. □

We observe that if the regions use only one of two weights $\{1, \alpha\}$, Corollary 2 implies that the time complexity of the algorithm proposed by Mitchell and Papadimitriou [15] can be improved by a quartic factor to $O(n^4 L)$, where L is the precision of the instance.

3 Computing a shortest path

We now consider the problem of computing a shortest path $\pi(s,t)$ from s to t when the region R is an axis-aligned rectangle of weight α. The exact shape of $\pi(s,t)$ depends on the position of s and t with respect to R, and on the value of α. In Sections 3.1 and 3.2 we consider the case that s lies on the boundary or inside of R, respectively. We categorize the various types of shortest paths, and show that we can compute the shortest path of each type, and thus we can compute $\pi(s,t)$. In Section 3.3, we consider the case that s and t lie outside R. In this case $\pi(s,t)$ may have only two vertices on the boundary of R, and these vertices may not have the critical angle property. We show that the coordinates of these vertices cannot be computed exactly within the ACM$\mathbb{Q}$.

3.1 The source point s lies on the boundary of R

Throughout this section we consider the case where s is restricted to the boundary of R, a rectangle of unit height with top-left corner at $(0,0)$. Let $s = (s_x, 0)$, $s_x > 0$, be a point on the top side of R, see Figure 2. In addition, we assume that t is to the left of the line through s perpendicular to the top side of R. The other cases are symmetric.

Shortest path types. Lemma 1 implies that in this setting, there are only $O(1)$ combinatorial types of paths that we have to consider. More precisely, we have that:

Observation 1 *Let s be a point on the top boundary of a rectangle R with weight $0 < \alpha < \sqrt{2}$. There are 12 types of shortest paths $\pi_i(s,t)$, shown in Figure 2, up to symmetries.*

Note that only some of the types can exist for both $\alpha < 1$ and $1 \leq \alpha < \sqrt{2}$. These types are included twice in Figure 2, once for each regime of α.

Length of $\pi_i(s,t)$. When s is on the boundary of R, there is at most one vertex of $\pi_i(s,t)$ without the critical angle property. This allows us to compute the exact coordinates of the vertices of $\pi_i(s,t)$ in the ACM$\mathbb{Q}$. We now provide the equations for the length $d_i(s,t)$ of the 12 types of shortest paths $\pi_i(s,t)$. Theorems 3 and 4 summarize the results. The proofs of the equations, which are based on Snell's law of refraction, are deferred to Appendix A.

Theorem 3 *Let $s = (s_x, 0)$ be a point on the boundary of R with weight $0 < \alpha < \sqrt{2}$, and let $\beta = \alpha^2 - 1$. The shortest path $\pi(s,t) = \pi_i(s,t)$ from s to a point $t = (t_x, t_y)$ outside R, and its length can be computed in $O(1)$ time in the ACM$\mathbb{Q}$. In particular, the length $d(s,t) = d_i(s,t)$ is given by*

- $d_1(s,t) = \sqrt{(s_x - t_x)^2 + t_y^2}$,
- $d_2(s,t) = \alpha(s_x - t_x) + \sqrt{1-\alpha^2}\, t_y$,
- $d_3(s,t) = \alpha s_x + \sqrt{t_x^2 + t_y^2}$,
- $d_4(s,t) = s_x + \sqrt{t_x^2 + t_y^2}$,
- $d_5(s,t) = s_x - \sqrt{1-\beta}\, t_x - \sqrt{\beta}\, t_y$,
- $d_6(s,t) = \alpha\sqrt{s_x^2 + y^2} + \sqrt{t_x^2 + (t_y - y)^2}$, *where y is the unique real solution in the interval $(t_y, 0)$ to the equation*

$$\beta y^4 - 2t_y\beta y^3 + \left[\alpha^2 t_x^2 + \beta t_y^2 - s_x^2\right] y^2 + 2s_x^2 t_y y - s_x^2 t_y^2 = 0,$$

- $d_7(s,t) = \sqrt{\beta}\, s_x + 1 + \sqrt{t_x^2 + (t_y + 1)^2}$,
- $d_8(s,t) = \sqrt{\beta}(s_x + t_x) - \sqrt{1-\beta}(1 + t_y) + 1$,
- $d_9(s,t) = \alpha\sqrt{(s_x - x)^2 + 1} + \sqrt{(t_x - x)^2 + (t_y + 1)^2}$, *where x is the unique real solution in the interval (t_x, s_x) to the equation*

$$\begin{aligned} &\beta x^4 - 2\beta(t_x + s_x)x^3 + \big[\beta(s_x^2 + t_x^2 + 4s_x t_x) \\ &\quad + \alpha^2(1+t_y)^2 - 1\big]x^2 - 2\big[\beta(t_x s_x^2 + t_x^2 s_x) \\ &\quad + \alpha^2(1+t_y)^2 s_x - t_x\big]x + \beta t_x^2 s_x^2 \\ &\quad + \alpha^2(1+t_y)^2 s_x^2 - t_x^2 = 0. \end{aligned} \tag{1}$$

Theorem 4 *Let $s = (s_x, 0)$ be a point on the boundary of R with weight $0 < \alpha < \sqrt{2}$. The shortest path $\pi(s,t) = \pi_i(s,t)$ from s to a point $t = (t_x, t_y)$ inside R, and its length can be computed in $O(1)$ time in the ACM$\mathbb{Q}$. In particular, the length $d(s,t) = d_i(s,t)$ is given by*

- $d_{10}(s,t) = s_x - t_x - \sqrt{\alpha^2 - 1}\, t_y$,
- $d_{11}(s,t) = \alpha\sqrt{(s_x - t_x)^2 + t_y^2}$,
- $d_{12}(s,t) = \sqrt{\alpha^2 - 1}(s_x + t_x) - t_y$.

3.2 The source point s lies inside R

We now consider the case where s is restricted to the interior of the rectangle R.

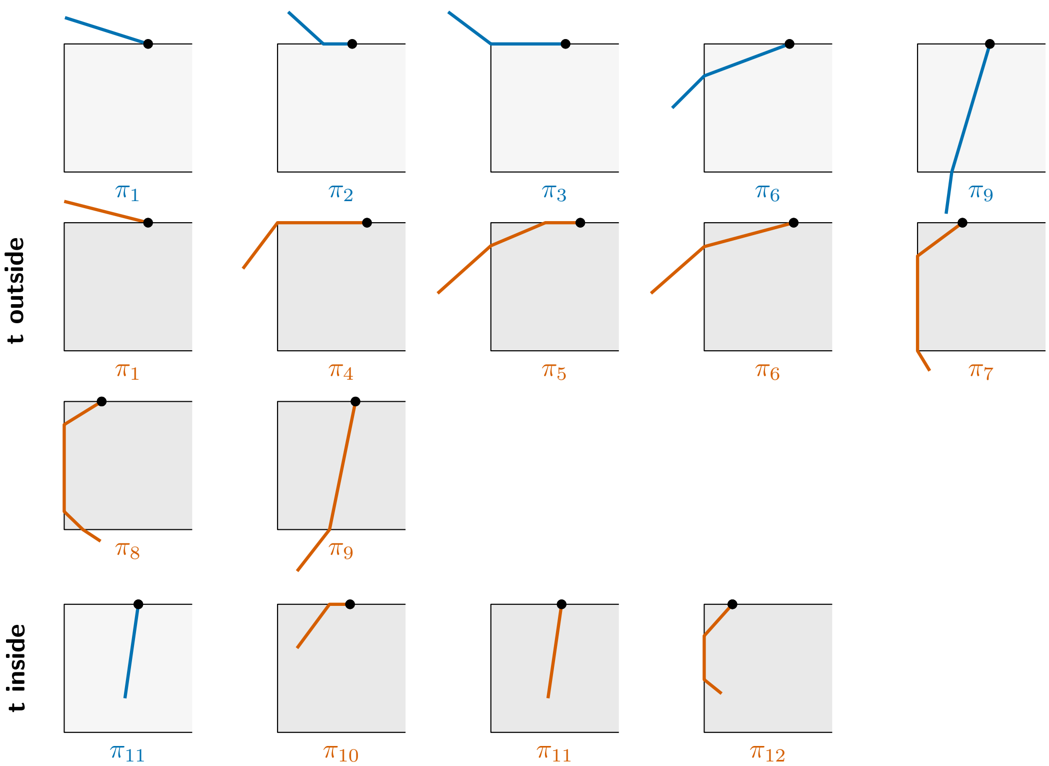

Figure 2: Path types for s on the boundary of R of weight $\alpha < 1$ (blue) and $1 \leq \alpha < \sqrt{2}$ (orange).

Observation 2 *Let s be a point in a rectangle R with weight $0 < \alpha < \sqrt{2}$. There are 6 types of shortest paths $\pi_i(s,t)$, for $i \in \{6, 7, 8, 9, 11, 12\}$, up to symmetries.*

The types of shortest paths are similar to the ones defined in Observation 1, see the paths in Figure 2 where the top side of R or the region above R is not intersected. As in Theorems 3 and 4, we can thus compute (the length of) a shortest path (of each type) exactly, albeit that the expressions for the length are dependent on the location of s in R. Note that Theorems 3 and 4 give exact lengths for all path types when R has height > 1 and s is at distance exactly 1 from the bottom boundary of R.

3.3 The source point s lies outside of R

When both the source and the target point are outside of R, the shortest path can again be of many different types. In particular, the types in Figure 2 can be generalized to this setting. There are two special cases where the shortest path bends *twice*, and these two vertices do not have the critical angle property: it can bend on two opposite sides of the rectangle, or on two incident sides. In the first case, the angles at both vertices are equal, and the shortest path can be computed exactly [16]. For the second case, we show that it is not possible to compute the coordinates of the vertices exactly in the ACM$\mathbb{Q}$. Hence, the WRP limited to two weights $\{1, \alpha\}$ is not solvable within the ACM$\mathbb{Q}$. Note that this path type can occur in an even simpler setting, where R is a single quadrant instead of a rectangle.

Theorem 5 *The Weighted Region Problem with weights in the set $\{1, \alpha\}$, with $0 < \alpha < \sqrt{2}$, and $\alpha \neq 1$, cannot be solved exactly within the ACM$\mathbb{Q}$, even if R is a single quadrant.*

Proof. Consider the situation where a horizontal and a vertical line intersect at the point $O = (50, 150)$. Let R be the quadrant such that O is its top-left corner, and has weight $\alpha = 1.2$. Recall that the weight outside R is 1. Let $s = (0, 0)$ be the source point and $t = (200, 200)$ be the target point, see Figure 3. We follow the approach of De Carufel et al. [6] to show that the polynomial that represents a solution to the Weighted Region Problem in this situation is not solvable within the ACM$\mathbb{Q}$. The following lemma, which is a consequence of Theorem 1 and Lemma 2 of De Carufel et al. [6], see also [4, 7], states when a polynomial is unsolvable within the ACM$\mathbb{Q}$.

Lemma 6 *Let $p(x)$ be a polynomial of odd degree $d \geq 5$. Suppose there are three primes q_1, q_2, q_3 that do not divide the discriminant of $p(x)$, such that*

$$\begin{aligned} p(x) &\equiv p_d(x) \mod q_1, \\ p(x) &\equiv p_1(x)p_{d-1}(x) \mod q_2, \text{ and} \\ p(x) &\equiv p_2(x)p_{d-2}(x) \mod q_3, \end{aligned}$$

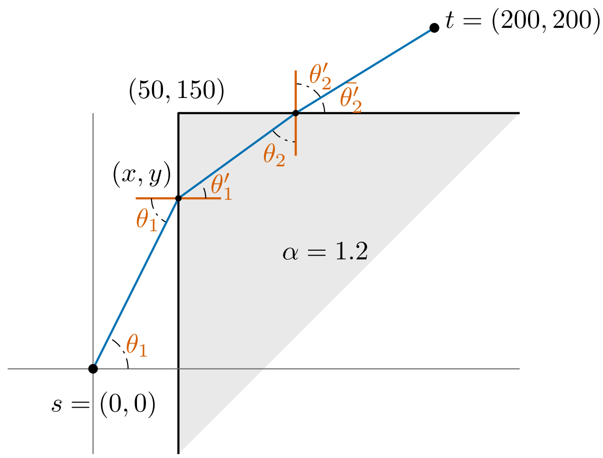

Figure 3: The set-up for the proof that even for two weights the Weighted Region Problem cannot be solved within the ACMℚ.

where $p_i(x)$ denotes an irreducible polynomial of degree i modulo the given prime. Then $p(x) = 0$ is unsolvable within the ACMℚ.

Let (x, y) be the coordinates of the intersection point of the path $\pi(s, t)$ with the vertical side of the quadrant. We denote by θ_1 the angle made by the ray from s to (x, y) with the perpendicular to the vertical side of the quadrant, by θ_1' the angle of the refracted ray with respect to the same line, and by $\bar{\theta}_2'$ the angle of the refracted ray with respect to the top side of the quadrant, see Figure 3. First we use Snell's law to show the following relations between the angles:

$$\sin\theta_1' = \frac{\sin\theta_1}{\alpha}, \tag{2}$$

$$\cos\bar{\theta}_2' = \sqrt{\alpha^2 - \sin^2\theta_1}. \tag{3}$$

To obtain the relation in Equation (3) we use Equation (2), and the fact that $\cos\bar{\theta}_2' = \alpha\cos\theta_1'$, and $\cos\theta_1' = \sqrt{1 - \sin^2\theta_1'}$. We then express the sum of the horizontal distances in terms of tangents of the angles, as follows:

$$200 = 50 + \frac{150 - y}{\tan\theta_1'} + \frac{50}{\tan\bar{\theta}_2'}$$
$$\implies 0 = \frac{150 - y}{\tan\theta_1'} + \frac{50}{\tan\bar{\theta}_2'} - 150.$$

Using that $y = 50\tan\theta_1$, we obtain an equation only containing θ_1, θ_1' and $\bar{\theta}_2'$.

$$0 = \frac{150 - 50\tan\theta_1}{\tan\theta_1'} + \frac{50}{\tan\bar{\theta}_2'} - 150.$$

We then apply the trigonometric identities $\tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$, for θ_1 and θ_1', and $\tan\theta = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$, for $\bar{\theta}_2'$.

$$0 = \frac{150 - 50\frac{\sin\theta_1}{\sqrt{1-\sin^2\theta_1}}}{\frac{\sin\theta_1'}{\sqrt{1-\sin^2\theta_1'}}} + \frac{50}{\frac{\sqrt{1-\cos^2\bar{\theta}_2'}}{\cos\bar{\theta}_2'}} - 150.$$

Finally, we replace all instances of $\sin\theta_1'$ and $\cos\bar{\theta}_2'$ by expressions in $\sin\theta_1$ using Equations (2) and (3).

$$\begin{aligned}
0 &= \frac{150 - 50\frac{\sin\theta_1}{\sqrt{1-\sin^2\theta_1}}}{\frac{\sin\theta_1}{\alpha\sqrt{1-\frac{\sin^2\theta_1}{\alpha^2}}}} + \frac{50}{\frac{\sqrt{1-(\alpha^2-\sin^2\theta_1)}}{\sqrt{\alpha^2-\sin^2\theta_1}}} - 150 \\
&= \left(150 - 50\frac{\sin\theta_1}{\sqrt{1-\sin^2\theta_1}}\right) \cdot \frac{\sqrt{\alpha^2 - \sin^2\theta_1}}{\sin\theta_1} \\
&\quad + \frac{50\sqrt{\alpha^2 - \sin^2\theta_1}}{\sqrt{1 - (\alpha^2 - \sin^2\theta_1)}} - 150 \\
&= 50\sqrt{\alpha^2 - \sin^2\theta_1}\left(\frac{3}{\sin\theta_1} - \frac{1}{\sqrt{1 - \sin^2\theta_1}}\right. \\
&\quad \left. + \frac{1}{\sqrt{1 - \alpha^2 + \sin^2\theta_1}}\right) - 150.
\end{aligned}$$

The final equation in terms of $u = \sin\theta_1$ then becomes

$$\sqrt{\alpha^2 - u^2}\left(\frac{3}{u} - \frac{1}{\sqrt{1-u^2}} + \frac{1}{\sqrt{1-\alpha^2+u^2}}\right) = 3.$$

For $\alpha = 1.2$, this can be transformed into the following polynomial by squaring appropriately using Mathematica [20]:

$$\begin{aligned}
p(u) = &- 5602195930320001 + 93511401766200000u \\
&- 713160370741499900u^2 + 3259398736514250000u^3 \\
&- 9869397269940000000u^4 + 20717559301050000000u^5 \\
&- 30701172521250000000u^6 + 32082903984375000000u^7 \\
&- 23159988281250000000u^8 + 10999072265625000000u^9 \\
&- 3093750000000000000u^{10} + 390625000000000000u^{11}.
\end{aligned}$$

To show that polynomial $p(u)$ is unsolvable, we thus need three primes q_1, q_2, q_3 that adhere to the conditions in Lemma 6. Using Mathematica we find the following expressions for $p(u)$ modulo 59, 37, and 17, respectively:

$$\begin{aligned}
46\big(&u^{11} + 44u^{10} + 32u^9 + 33u^8 + 26u^7 + 47u^6 + 21u^5 \\
&+ 11u^4 + 38u^3 + 3u^2 + 6u + 42\big),
\end{aligned}$$

$$\begin{aligned}
16(u + 17)\big(&u^{10} + 18u^9 + 23u^8 + 23u^7 + 35u^6 + 8u^5 \\
&+ 34u^4 + 16u^3 + 11u^2 + 34u + 10\big),
\end{aligned}$$

$$4\big(u^2+14u+9\big)\big(u^9+8u^8+11u^7+3u^6+5u^5 + 2u^4+2u^3+12u^2+9u+16\big).$$

We conclude that even the very limited weighted region problem where we allow for a single quadrant to have weight unequal to 1 and s and t are on halfplanes bounded by the sides of the quadrant, not containing the quadrant, is not solvable within the ACM$\mathbb{Q}$. □

4 Computing a Shortest Path Map

To find a shortest path from a source point s to *all* points at once, one can build a *Shortest Path Map* (*SPM*), see e.g., [10, 14, 15]. A *SPM* is a subdivision of the space for a given source s, where for each cell the paths $\pi(s,t)$, with t in the cell, have the same type. With it, we are able to find for each specific destination t, the weight of the shortest path from s to t simply by locating the point t in the subdivision. Once a *SPM* is available, we are able to report weights of shortest paths from s to any destination t by standard point location techniques [8, 12]. To compute the *SPM*, we consider computing the bisectors $b_{i,j} = \{q \mid q \in \mathbb{R}^2 \wedge d_i(s,q) = d_j(s,q)\}$ for all relevant pairs of shortest path types π_i, π_j, i.e., pairs for which $b_{i,j}$ appears in the Shortest Path Map. A *SPM* requires only polynomial space. However, in general, the bisector curves that bound cells of the *SPM* subdivision will be curves of very high degree [6, 11].

As before, we consider the setting where R is a rectangular region. In Section 4.1, we first consider the case when s lies on the boundary of R. In Section 4.2, we do the same for the case s lies inside R. The case that s lies outside R is not interesting, as we cannot even compute exactly a single shortest path in that case.

4.1 The source point s lies on the boundary of R

The *SPM* is given by the boundary of R and several bisector curves, expressed as points $(x, b_{i,j}(x))$. If $\alpha < 1$, these curves all lie outside R (the interior of R is a single region in the *SPM*).

Furthermore, bisectors involving $\pi_9(s,t)$ are of a much more complicated form, as might be expected from the implicit representation used for $d_9(s,t)$ in Theorem 3. Therefore, Lemmas 7 and 8 give the bisector curves, excluding the ones related to $\pi_9(s,t)$. The proofs are deferred to Appendix B.

Lemma 7 *The SPM for a point $s = (s_x, 0)$ on the boundary of the region R with weight $\alpha < 1$ is defined by:*

$$b_{i,j}(x) = \begin{cases} \frac{\sqrt{1-\alpha^2}}{\alpha}(s_x - x) & \text{if } i=1, j=2 \\ -\frac{\sqrt{1-\alpha^2}}{\alpha}x & \text{if } i=2, j=3 \\ 0 & \text{if } i=3, j=6 \end{cases}$$

Lemma 8 *The SPM for a point $s = (s_x, 0)$ on the boundary of the region R with weight $1 < \alpha < \sqrt{2}$ is defined by:*

$$b_{i,j}(x) = \begin{cases} 0 & \text{if } i=1, j=4 \\ \frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}}x & \text{if } i=4, j=5 \\ \frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}}x - \sqrt{\alpha^2-1}s_x & \text{if } i=5, j=6 \\ x=0 & \text{if } i=6, j=7 \\ -1-\frac{\sqrt{2-\alpha^2}}{\sqrt{\alpha^2-1}}x & \text{if } i=7, j=8 \\ -\sqrt{\alpha^2-1}(s_x - x) & \text{if } i=10, j=11 \\ -\frac{(s_x+x)+2\alpha\sqrt{s_x x}}{\sqrt{\alpha^2-1}} & \text{if } i=11, j=12 \end{cases}$$

We conjecture the following on the bisectors involving $\pi_9(s,t)$.

Conjecture 1 *No point on $b_{i,9}(x) \setminus R$, $i \in \{4,\ldots,8\}$, can be computed exactly within ACM$\mathbb{Q}$.*

We tried to prove this conjecture by taking a similar approach as in Theorem 5. However, the solution to Equation (1) already seems to be of high degree. We therefore did not manage to formulate a point on the bisector as a polynomial equation (not containing roots).

Note that in the more restrictive case where R is a single quadrant and s lies on its boundary, the only types of shortest paths that exist are $\pi_i(s,t)$, for $i \in \{1,2,3,4,5,6,10,11,12\}$. Thus, we can compute the *SPM* in the ACM$\mathbb{Q}$ (the bisectors are given by some of the equations in Lemmas 7 and 8).

4.2 The source point s lies inside R

In this case we have shortest paths of type $\pi_i(s,t)$, for $i \in \{6,7,8,9,11,12\}$. Hence, the equations of the bisectors of the *SPM* are given by the sides of R, and bisector $b_{6,9}$ if $\alpha < 1$, and bisectors $b_{6,7}, b_{7,8}, b_{6,9}, b_{7,9}, b_{8,9}$ and $b_{11,12}$ if $1 < \alpha < \sqrt{2}$. See Lemmas 7 and 8, and Conjecture 1.

5 Conclusion

We analyzed the WRP when there is only one weighted rectangle R, and showed how to obtain the exact shortest path $\pi(s,t)$ and its length when s lies in or on R. When both s and t lie outside R the exact solution is unsolvable in the ACM$\mathbb{Q}$. We obtain similar results in the case where R is a single quadrant. For future work, it would be interesting to find an exact formula within the ACM$\mathbb{Q}$ for the bisectors involving $\pi_9(s,t)$. In addition, we may want to analyze if or how we can generalize these results to other convex shapes.

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A Computing a shortest path when the source point s lies on the boundary of R

In this section, we prove the stated lengths of the shortest paths as defined in Theorems 3 and 4. All angles in this section are measured relative to the vertical or horizontal line through the bending point for the bending point on the top (and bottom) or left side of R, respectively. We denote by θ_c the critical angle. The following two equations capture the properties we use of the critical angle. For $\alpha < 1$, we have

$$\sin\theta_c = \alpha \Rightarrow \begin{cases} \cos\theta_c = \sqrt{1-\alpha^2} \\ \tan\theta_c = \frac{\sin\theta_c}{\cos\theta_c} = \frac{\alpha}{\sqrt{1-\alpha^2}}. \end{cases} \tag{4}$$

And for $\alpha > 1$, we have

$$\sin\theta_c = \frac{1}{\alpha} \Rightarrow \begin{cases} \cos\theta_c = \sqrt{1-\frac{1}{\alpha^2}} \\ \tan\theta_c = \frac{\sin\theta_c}{\cos\theta_c} = \frac{\frac{1}{\alpha}}{\sqrt{1-\frac{1}{\alpha^2}}} = \frac{1}{\sqrt{\alpha^2-1}}. \end{cases} \tag{5}$$

We frequently use these equations in the rest of this section to determine the lengths of the path $d_i(s,t)$ for all i.

From now on, we consider the case where the source point s is restricted to the boundary of R, an axis-aligned rectangle of unit height with top-left corner at $(0,0)$.

Observation 3 *Let R be a rectangular region with weight $0 < \alpha < \sqrt{2}$. Let $s = (s_x, 0)$ be a point on the boundary of R. Then the length of the shortest path $\pi_1(s,t)$ from s to a point $t = (t_x, t_y)$ outside R is given by $d_1(s,t) = \sqrt{(s_x - t_x)^2 + t_y^2}$.*

Lemma 9 *Let R be a rectangular region with weight $0 < \alpha < 1$. Let $s = (s_x, 0)$ be a point on the boundary of R. Then the length of the shortest paths $\pi_2(s,t)$ from s to a point $t = (t_x, t_y)$ outside R is given by $d_2(s,t) = \alpha(s_x - t_x) + \sqrt{1-\alpha^2}\, t_y$.*

Proof. Let θ_c be the critical angle made by $\pi_2(s,t)$ on the top boundary of R, and let $(b,0)$ be the point where the shortest path leaves R. We use Equation (4) to obtain the value of b:

$$\frac{\alpha}{\sqrt{1-\alpha^2}} = \frac{|b-t_x|}{|t_y|} = \frac{b-t_x}{t_y} \Rightarrow b = t_x + \frac{\alpha}{\sqrt{1-\alpha^2}} t_y. \tag{6}$$

We know that the weight of the shortest paths $\pi_2(s,t)$ is given by $d_2(s,t) = \alpha|s_x - b| + \sqrt{(b-t_x)^2 + t_y^2}$. By using Equation (6), we have that

$$\begin{aligned} d_2(s,t) &= \alpha\left(s_x - t_x - \frac{\alpha}{\sqrt{1-\alpha^2}} t_y\right) \\ &\quad + \sqrt{\left(t_x + \frac{\alpha}{\sqrt{1-\alpha^2}} t_y - t_x\right)^2 + t_y^2} \\ &= \alpha\left(s_x - t_x - \frac{\alpha}{\sqrt{1-\alpha^2}} t_y\right) + \sqrt{\frac{\alpha^2 t_y^2}{1-\alpha^2} + t_y^2} \\ &= \alpha(s_x - t_x) - \frac{\alpha^2 t_y}{\sqrt{1-\alpha^2}} + \sqrt{\frac{t_y^2}{1-\alpha^2}} \\ &= \alpha(s_x - t_x) - \frac{\alpha^2 t_y}{\sqrt{1-\alpha^2}} + \frac{|t_y|}{\sqrt{1-\alpha^2}} \\ &= \alpha(s_x - t_x) + \frac{1-\alpha^2}{\sqrt{1-\alpha^2}} t_y \\ &= \alpha(s_x - t_x) + \sqrt{1-\alpha^2}\, t_y. \end{aligned}$$

□

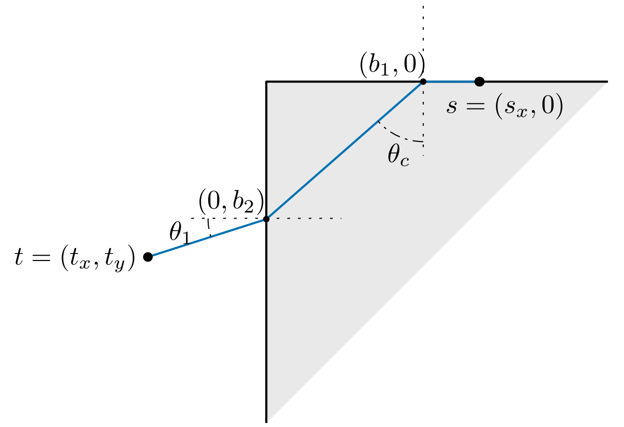

Figure 4: Illustration of Lemma 10.

Observation 4 *Let R be a rectangular region with weight $0 < \alpha < 1$. Let $s = (s_x, 0)$ be a point on the boundary of R. Then the length of the shortest paths $\pi_3(s,t)$ from s to a point $t = (t_x, t_y)$ outside R is given by $d_3(s,t) = \alpha s_x + \sqrt{t_x^2 + t_y^2}$.*

Observation 5 *Let R be a rectangular region with weight $1 < \alpha < \sqrt{2}$. Let $s = (s_x, 0)$ be a point on the boundary of R. Then the length of the shortest path $\pi_4(s,t)$ from s to a point $t = (t_x, t_y)$ outside R is given by $d_4(s,t) = s_x + \sqrt{t_x^2 + t_y^2}$.*

Lemma 10 *Let R be a rectangular region with weight $1 < \alpha < \sqrt{2}$. Let $s = (s_x, 0)$ be a point on the boundary of R. Then the length of the shortest path $\pi_5(s,t)$ from s to a point $t = (t_x, t_y)$ outside R is given by $d_5(s,t) = s_x - \sqrt{2-\alpha^2}\, t_x - \sqrt{\alpha^2 - 1}\, t_y$.*

Proof. The shortest path from s to t intersects the top side of R, and then it enters R using the critical angle, see Figure 4. We proceed to compute the coordinates of the vertices of the shortest path in this case.

Let θ_1 be the angle at which the shortest path leaves R with respect to the normal, see Figure 4. Then

$$\sin\theta_1 = \alpha \sin\left(\frac{\pi}{2} - \theta_c\right) = \alpha\cos\theta_c = \sqrt{\alpha^2 - 1},$$

and thus

$$\tan\theta_1 = \frac{\sin\theta_1}{\cos\theta_1} = \frac{\sqrt{\alpha^2-1}}{\sqrt{1-(\alpha^2-1)}} = \frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}}. \tag{7}$$

Let $(b_1, 0)$ and $(0, b_2)$ be, respectively, the points where the shortest path enters and leaves R. We also know that $\tan\theta_1 = \frac{|t_y - b_2|}{|t_x|}$. Since $t_y < b_2 < 0$, and $t_x < 0$, we use Equation (7) to get the value of b_2:

$$\begin{aligned} \frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}} &= \frac{|t_y - b_2|}{|t_x|} = \frac{b_2 - t_y}{-t_x} \\ &\Rightarrow b_2 = t_y - \frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}} t_x. \end{aligned} \tag{8}$$

Also, since $\tan\theta_c = \frac{|b_1|}{|b_2|}$, and $b_2 < 0 < b_1$, we use Equation (5) to get the value of b_1:

$$\frac{1}{\sqrt{\alpha^2-1}} = \frac{|b_1|}{|b_2|} = \frac{b_1}{-b_2}$$

$$\Rightarrow b_1 = -\frac{b_2}{\sqrt{\alpha^2-1}} = -\frac{t_y}{\sqrt{\alpha^2-1}} + \frac{t_x}{\sqrt{2-\alpha^2}}.$$

Since $\sin\theta_c = \frac{b_1}{\sqrt{b_1^2+b_2^2}} = \frac{1}{\alpha} \Rightarrow \sqrt{b_1^2+b_2^2} = b_1\alpha$. Thus:

$$\begin{aligned}
d_5(s,t) &= s_x - b_1 + \alpha\sqrt{b_1^2+b_2^2} + \sqrt{t_x^2+(t_y-b_2)^2} \\
&= s_x - b_1 + b_1\alpha^2 + \sqrt{t_x^2 + \left(t_y - t_y + \frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}}t_x\right)^2} \\
&= s_x + (\alpha^2-1)b_1 + \sqrt{t_x^2 + \frac{\alpha^2-1}{2-\alpha^2}t_x^2} \\
&= s_x + (\alpha^2-1)b_1 + |t_x|\sqrt{1+\frac{\alpha^2-1}{2-\alpha^2}} \\
&= s_x + (\alpha^2-1)b_1 + |t_x|\sqrt{\frac{2-\alpha^2+\alpha^2-1}{2-\alpha^2}} \\
&= s_x + (\alpha^2-1)\left(\frac{t_x}{\sqrt{2-\alpha^2}} - \frac{t_y}{\sqrt{\alpha^2-1}}\right) - t_x\frac{1}{\sqrt{2-\alpha^2}} \\
&= s_x - \sqrt{\alpha^2-1}t_y + \frac{\alpha^2-1-1}{\sqrt{2-\alpha^2}}t_x \\
&= s_x - \sqrt{\alpha^2-1}t_y - \frac{2-\alpha^2}{\sqrt{2-\alpha^2}}t_x \\
&= s_x - \sqrt{\alpha^2-1}t_y - \sqrt{2-\alpha^2}t_x. \qquad \square
\end{aligned}$$

Lemma 11 *Let R be a rectangular region with weight $0 < \alpha < \sqrt{2}$. Let $s = (s_x, 0)$ be a point on the boundary of R. Then the length of the shortest paths $\pi_6(s,t)$ from s to a point $t = (t_x, t_y)$ outside R is given by $d_6(s,t) = \alpha\sqrt{s_x^2+y^2} + \sqrt{t_x^2+(t_y-y)^2}$, where y is the unique real solution in the interval $(t_y, 0)$ to the equation*

$$(\alpha^2-1)y^4 - 2t_y(\alpha^2-1)y^3 + [\alpha^2t_x^2 + (\alpha^2-1)t_y^2 - s_x^2]y^2 + 2s_x^2t_yy - s_x^2t_y^2 = 0,$$

Proof. Let $(0,y)$ be the point where $\pi_6(s,t)$ leaves R, and let θ_1 and θ_2 be, respectively, the angles of incidence and refraction at $(0,y)$. Then, by Snell's law of refraction, we get that $\alpha\sin\theta_1 = \sin\theta_2$. Thus,

$$\alpha\frac{|y|}{\sqrt{s_x^2+y^2}} = \frac{|t_y-y|}{\sqrt{t_x^2+(t_y-y)^2}}$$
$$\Rightarrow \alpha^2y^2(t_x^2+(t_y-y)^2) = (t_y-y)^2(s_x^2+y^2)$$
$$\Rightarrow \alpha^2y^2t_x^2 + \alpha^2y^2t_y^2 + \alpha^2y^4 - 2\alpha^2y^3t_y = s_x^2t_y^2 + s_x^2y^2 - 2s_x^2t_yy + y^2t_y^2 + y^4 - 2y^3t_y.$$

Hence,

$$(\alpha^2-1)y^4 - 2t_y(\alpha^2-1)y^3 + [\alpha^2t_x^2 + (\alpha^2-1)t_y^2 - s_x^2]y^2 + 2s_x^2t_yy - s_x^2t_y^2 = 0$$

Finally, we get that the weighted length of the shortest paths $\pi_6(s,t)$ is given by

$$d_6(s,t) = \alpha\sqrt{s_x^2+y^2} + \sqrt{t_x^2+(t_y-y)^2}. \qquad \square$$

Lemma 12 *Let R be a rectangular region with weight $1 < \alpha < \sqrt{2}$. Let $s = (s_x, 0)$ be a point on the boundary of R. Then the length of the shortest paths $\pi_7(s,t)$ from s to a point $t = (t_x, t_y)$ outside R is given by $d_7(s,t) = \sqrt{\alpha^2-1}s_x + 1 + \sqrt{t_x^2+(t_y+1)^2}$.*

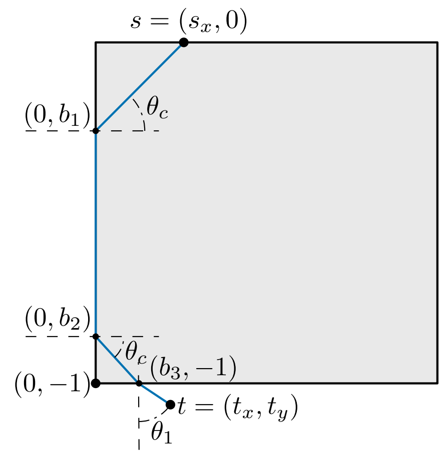

Figure 5: Illustration of Lemma 13.

Proof. Let $(0, b_1)$ be the point where $\pi_7(s,t)$ leaves R, i.e., the first vertex of the shortest path. Since $b_1 < 0$, and $s_x > 0$, we obtain the coordinates of this first bending point by using Equation (5):

$$\tan\theta_c = \frac{|b_1|}{|s_x|} = \frac{-b_1}{s_x} = \frac{1}{\sqrt{\alpha^2-1}} \Rightarrow b_1 = -\frac{s_x}{\sqrt{\alpha^2-1}}. \quad (9)$$

The weight of the shortest paths $\pi_7(s,t)$ is then given by

$$\begin{aligned}
d_7(s,t) &= \alpha\sqrt{s_x^2+b_1^2} + (b_1+1) + \sqrt{t_x^2+(-1-t_y)^2} \\
&= \frac{\alpha^2s_x}{\sqrt{\alpha^2-1}} - \frac{s_x}{\sqrt{\alpha^2-1}} + 1 + \sqrt{t_x^2+(-1-t_y)^2} \\
&= \sqrt{\alpha^2-1}s_x + 1 + \sqrt{t_x^2+(t_y+1)^2}. \qquad \square
\end{aligned}$$

Lemma 13 *Let R be a rectangular region with weight $1 < \alpha < \sqrt{2}$. Let $s = (s_x, 0)$ be a point on the boundary of R. Then the length of the shortest path $\pi_8(s,t)$ from s to a point $t = (t_x, t_y)$ outside R is given by $d_8(s,t) = \sqrt{\alpha^2-1}(s_x + t_x) - \sqrt{2-\alpha^2}(1+t_y) + 1$.*

Proof. Let $(0, b_1)$ be the point where $\pi_8(s,t)$ leaves R for the first time and let $(0, b_2)$ and $(b_3, -1)$ be, respectively, the points where $\pi_8(s,t)$ enters and leaves R for the second time, see Figure 5. As $\pi_7(s,t)$ and $\pi_8(s,t)$ overlap up to b_2, Equation (9) gives us that $b_1 = -\frac{s_x}{\sqrt{\alpha^2-1}}$.

Recall that R has height 1. Let θ_1 be the angle at which the shortest path leaves R for the last time with respect to the normal, see Figure 5. Then, $\tan\theta_1 = \frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}}$, similar to Equation (7). So,

$$\tan\theta_1 = \frac{|t_x-b_3|}{|t_y+1|} = \frac{t_x-b_3}{-1-t_y} = \frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}}$$
$$\Rightarrow b_3 = t_x + (1+t_y)\frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}}.$$

And, using Equation (5), we get

$$\tan\theta_c = \frac{|-1-b_2|}{|b_3|} = \frac{b_2+1}{b_3} = \frac{1}{\sqrt{\alpha^2-1}}$$

$$\Rightarrow b_2 = \frac{b_3}{\sqrt{\alpha^2-1}} - 1 = \frac{t_x}{\sqrt{\alpha^2-1}} + \frac{1+t_y}{\sqrt{2-\alpha^2}} - 1.$$

The weight of the shortest paths $\pi_8(s,t)$ is given by $d_8(s,t) = \alpha\sqrt{s_x^2+b_1^2} + |b_2 - b_1| + \alpha\sqrt{b_3^2+(b_2+1)^2} + \sqrt{(b_3-t_x)^2+(-1-t_y)^2}$. Using the expression for b_1, we have that

$$\sqrt{s_x^2+b_1^2} = \sqrt{s_x^2 + \left(-\frac{s_x}{\sqrt{\alpha^2-1}}\right)^2} = \sqrt{\frac{\alpha^2-1+1}{\alpha^2-1}s_x^2} = \frac{\alpha s_x}{\sqrt{\alpha^2-1}}. \tag{10}$$

Using our expressions for b_2 and b_3, and the fact that $b_2+1>0$, we obtain the following for the terms $A = \sqrt{b_3^2+(b_2+1)^2}$ and $B = \sqrt{(b_3-t_x)^2+(-1-t_y)^2}$:

$$\begin{aligned} A &= \sqrt{\left(t_x + (1+t_y)\frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}}\right)^2 + \left(\frac{t_x}{\sqrt{\alpha^2-1}} + \frac{1+t_y}{\sqrt{2-\alpha^2}} - 1 + 1\right)^2} \\ &= \sqrt{(\alpha^2-1)\left(\frac{t_x}{\sqrt{\alpha^2-1}} + \frac{1+t_y}{\sqrt{2-\alpha^2}}\right)^2 + \left(\frac{t_x}{\sqrt{\alpha^2-1}} + \frac{1+t_y}{\sqrt{2-\alpha^2}}\right)^2} \\ &= \sqrt{\alpha^2\left(\frac{t_x}{\sqrt{\alpha^2-1}} + \frac{1+t_y}{\sqrt{2-\alpha^2}}\right)^2} \\ &= \alpha\left(\frac{t_x}{\sqrt{\alpha^2-1}} + \frac{1+t_y}{\sqrt{2-\alpha^2}}\right), \end{aligned} \tag{11}$$

and

$$\begin{aligned} B &= \sqrt{\left(t_x + (1+t_y)\frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}} - t_x\right)^2 + (-1-t_y)^2} \\ &= \sqrt{(1+t_y)^2\frac{\alpha^2-1}{2-\alpha^2} + (1+t_y)^2} \\ &= \sqrt{(1+t_y)^2\frac{\alpha^2-1+2-\alpha^2}{2-\alpha^2}} \\ &= \frac{|1+t_y|}{\sqrt{2-\alpha^2}}. \end{aligned} \tag{12}$$

Using Equations (10), (11), and (12), we get that the weighted length of the shortest paths $\pi_8(s,t)$ is given by

$$\begin{aligned} d_8(s,t) &= \frac{\alpha^2 s_x}{\sqrt{\alpha^2-1}} - \frac{s_x+t_x}{\sqrt{\alpha^2-1}} - \frac{1+t_y}{\sqrt{2-\alpha^2}} + 1 \\ &\quad + \alpha^2\left(\frac{t_x}{\sqrt{\alpha^2-1}} + \frac{1+t_y}{\sqrt{2-\alpha^2}}\right) + \frac{|1+t_y|}{\sqrt{2-\alpha^2}} \\ &= \alpha^2\frac{s_x+t_x}{\sqrt{\alpha^2-1}} - \frac{s_x+t_x}{\sqrt{\alpha^2-1}} + (\alpha^2-1)\frac{1+t_y}{\sqrt{2-\alpha^2}} \\ &\quad + 1 - \frac{1+t_y}{\sqrt{2-\alpha^2}} \\ &= (\alpha^2-1)\frac{s_x+t_x}{\sqrt{\alpha^2-1}} + (\alpha^2-2)\frac{1+t_y}{\sqrt{2-\alpha^2}} + 1 \\ &= \sqrt{\alpha^2-1}(s_x+t_x) - \sqrt{2-\alpha^2}(1+t_y) + 1. \quad \square \end{aligned}$$

Lemma 14 *Let R be a rectangular region with weight $0 < \alpha < \sqrt{2}$. Let $s = (s_x, 0)$ be a point on the boundary of R. Then the length of the shortest paths $\pi_9(s,t)$ from s to a point $t = (t_x, t_y)$ outside R is given by $d_9(s,t) = \alpha\sqrt{(s_x-x)^2+1} + \sqrt{(t_x-x)^2+(t_y+1)^2}$ where x is the unique real solution in the interval (t_x, s_x) to the equation*

$$\begin{aligned} &(\alpha^2-1)x^4 - 2(\alpha^2-1)(t_x+s_x)x^3 \\ &+ [(\alpha^2-1)(s_x^2+t_x^2+4s_xt_x) + \alpha^2(1+t_y)^2 - 1]x^2 \\ &- 2[(\alpha^2-1)(t_xs_x^2+t_x^2s_x) + \alpha^2(1+t_y)^2s_x - t_x]x \\ &+ (\alpha^2-1)t_x^2s_x^2 + \alpha^2(1+t_y)^2s_x^2 - t_x^2 = 0. \end{aligned}$$

Proof. Let $(x,-1)$ be the point where $\pi_9(s,t)$ leaves R, and let θ_1 and θ_2 be, respectively, the angles of incidence and refraction at $(x,-1)$. Then, by Snell's law of refraction, we get that:

$$\begin{aligned} &\alpha\sin\theta_1 = \sin\theta_2 \Rightarrow \alpha\frac{s_x-x}{\sqrt{(s_x-x)^2+1}} = \frac{x-t_x}{\sqrt{(x-t_x)^2+(-1-t_y)^2}} \\ &\Rightarrow \alpha^2(x-t_x)^2(s_x-x)^2 + \alpha^2(-1-t_y)^2(s_x-x)^2 \\ &\quad = (x-t_x)^2(s_x-x)^2 + (x-t_x)^2 \\ &\Rightarrow (\alpha^2-1)(x-t_x)^2(s_x-x)^2 + \alpha^2(-1-t_y)^2(s_x-x)^2 \\ &\quad - (x-t_x)^2 = 0 \\ &\Rightarrow [(\alpha^2-1)x^2 + (\alpha^2-1)t_x^2 - 2(\alpha^2-1)t_xx] \\ &\quad \cdot (s_x^2+x^2-2s_xx) + \alpha^2(-1-t_y)^2s_x^2 + \alpha^2(-1-t_y)^2x^2 \\ &\quad - 2\alpha^2(-1-t_y)^2s_xx - x^2 - t_x^2 + 2t_xx = 0 \\ &\Rightarrow (\alpha^2-1)x^2s_x^2 + (\alpha^2-1)x^4 - 2(\alpha^2-1)s_xx^3 \\ &\quad + (\alpha^2-1)t_x^2s_x^2 + (\alpha^2-1)t_x^2x^2 - 2(\alpha^2-1)t_x^2s_xx \\ &\quad - 2(\alpha^2-1)t_xs_x^2x - 2(\alpha^2-1)t_xx^3 + 4(\alpha^2-1)s_xt_xx^2 \\ &\quad + \alpha^2(-1-t_y)^2s_x^2 + \alpha^2(-1-t_y)^2x^2 \\ &\quad - 2\alpha^2(-1-t_y)^2s_xx - x^2 - t_x^2 + 2t_xx = 0. \end{aligned}$$

Hence,

$$\begin{aligned} &(\alpha^2-1)x^4 - 2(\alpha^2-1)(t_x+s_x)x^3 \\ &+ [(\alpha^2-1)(s_x^2+t_x^2+4s_xt_x) + \alpha^2(1+t_y)^2 - 1]x^2 \\ &- 2[(\alpha^2-1)(t_xs_x^2+t_x^2s_x) + \alpha^2(1+t_y)^2s_x - t_x]x \\ &+ (\alpha^2-1)t_x^2s_x^2 + \alpha^2(1+t_y)^2s_x^2 - t_x^2 = 0. \end{aligned}$$

Finally, we get that the weighted length of the shortest paths $\pi_9(s,t)$ is given by

$$d_9(s,t) = \alpha\sqrt{(s_x-x)^2+1} + \sqrt{(t_x-x)^2+(1+t_y)^2}. \quad \square$$

Lemma 15 *Let R be a rectangular region with weight $1 < \alpha < \sqrt{2}$. Let $s = (s_x, 0)$ be a point on the boundary of R. Then the length of the shortest paths $\pi_{10}(s,t)$ from s to a point $t = (t_x, t_y)$ inside R is given by $d_{10}(s,t) = s_x - t_x - \sqrt{\alpha^2-1}t_y$.*

Proof. Let $(b_1, 0)$ be the point where $\pi_{10}(s,t)$ enters R. Let θ_c be the angle at which $\pi_{10}(s,t)$ enters R. Since θ_c is the critical angle, using Equation (5), we get the value of b_1:

$$\begin{aligned} \tan\theta_c &= \frac{|b_1-t_x|}{|t_y|} = \frac{b_1-t_x}{-t_y} = \frac{1}{\sqrt{\alpha^2-1}} \\ &\Rightarrow b_1 = t_x - \frac{t_y}{\sqrt{\alpha^2-1}}. \end{aligned} \tag{13}$$

We know that the weight of the shortest paths $\pi_{10}(s,t)$ is given by $d_{10}(s,t) = |s_x - b_1| + \alpha\sqrt{(b_1 - t_x)^2 + t_y^2}$. By using Equation (13), we have that

$$\begin{aligned}
d_{10}(s,t) = &\left(s_x - t_x + \frac{t_y}{\sqrt{\alpha^2-1}}\right) \\
&+ \alpha\sqrt{\left(t_x - \frac{t_y}{\sqrt{\alpha^2-1}} - t_x\right)^2 + t_y^2} \\
= &\left(s_x - t_x + \frac{t_y}{\sqrt{\alpha^2-1}}\right) + \alpha\sqrt{\frac{t_y^2}{\alpha^2-1} + t_y^2} \\
= &\ s_x - t_x + \frac{t_y}{\sqrt{\alpha^2-1}} + \alpha\sqrt{\frac{\alpha^2 t_y^2}{\alpha^2-1}} \\
= &\ s_x - t_x + \frac{t_y}{\sqrt{\alpha^2-1}} + \frac{\alpha^2|t_y|}{\sqrt{\alpha^2-1}} \\
= &\ s_x - t_x + \frac{t_y}{\sqrt{\alpha^2-1}} - \frac{\alpha^2 t_y}{\sqrt{\alpha^2-1}} \\
= &\ s_x - t_x - \frac{(\alpha^2-1)t_y}{\sqrt{\alpha^2-1}} \\
= &\ s_x - t_x - \sqrt{\alpha^2-1}\,t_y. \qquad \square
\end{aligned}$$

Observation 6 *Let R be a rectangular region with weight $0 < \alpha < \sqrt{2}$. Let $s = (s_x, 0)$ be a point on the boundary of R. Then the length of the shortest paths $\pi_{11}(s,t)$ from s to a point $t = (t_x, t_y)$ inside R is given by $d_{11}(s,t) = \alpha\sqrt{(s_x - t_x)^2 + t_y^2}$.*

Lemma 16 *Let R be a rectangular region with weight $1 < \alpha < \sqrt{2}$. Let $s = (s_x, 0)$ be a point on the boundary of R. Then the length of the shortest path $\pi_{12}(s,t)$ from s to a point $t = (t_x, t_y)$ inside R is given by $d_{12}(s,t) = \sqrt{\alpha^2-1}(s_x + t_x) - t_y$.*

Proof. Let $(0, b_1)$ and $(0, b_2)$ be the points where $\pi_{12}(s,t)$ leaves and enters for the second time, respectively, the region R. From Lemma 13 we know that $b_1 = -\frac{s_x}{\sqrt{\alpha^2-1}}$. Using Equation (5), we find:

$$\begin{aligned}
\tan\theta_c = \frac{|t_y - b_2|}{|t_x|} = \frac{b_2 - t_y}{t_x} &= \frac{1}{\sqrt{\alpha^2-1}} \\
\Rightarrow b_2 &= \frac{t_x}{\sqrt{\alpha^2-1}} + t_y.
\end{aligned}$$

We then get that the weight of $\pi_{12}(s,t)$ is given by $d_{12}(s,t) = \alpha\sqrt{s_x^2 + b_1^2} + |b_2 - b_1| + \alpha\sqrt{t_x^2 + (b_2 - t_y)^2}$. So:

$$\begin{aligned}
d_{12}(s,t) = &\ \alpha\sqrt{s_x^2 + \frac{s_x^2}{\alpha^2-1}} - \frac{s_x}{\sqrt{\alpha^2-1}} - \frac{t_x}{\sqrt{\alpha^2-1}} - t_y \\
&+ \alpha\sqrt{t_x^2 + \left(\frac{t_x}{\sqrt{\alpha^2-1}} + t_y - t_y\right)^2} \\
= &\ \alpha\sqrt{\frac{\alpha^2 s_x^2}{\alpha^2-1}} - \frac{s_x + t_x}{\sqrt{\alpha^2-1}} - t_y + \alpha\sqrt{t_x^2 + \frac{t_x^2}{\alpha^2-1}} \\
= &\ \frac{\alpha^2|s_x|}{\sqrt{\alpha^2-1}} - \frac{s_x + t_x}{\sqrt{\alpha^2-1}} - t_y + \alpha\sqrt{\frac{\alpha^2 t_x^2}{\alpha^2-1}} \\
= &\ \frac{\alpha^2 s_x}{\sqrt{\alpha^2-1}} - \frac{s_x + t_x}{\sqrt{\alpha^2-1}} - t_y + \frac{\alpha^2 t_x}{\sqrt{\alpha^2-1}} \\
= &\ \frac{\alpha^2(s_x + t_x)}{\sqrt{\alpha^2-1}} - \frac{s_x + t_x}{\sqrt{\alpha^2-1}} - t_y \\
= &\ \frac{\alpha^2 - 1}{\sqrt{\alpha^2-1}}(s_x + t_x) - t_y \\
= &\ \sqrt{\alpha^2-1}(s_x + t_x) - t_y. \qquad \square
\end{aligned}$$

B Computing the shortest path map for a given s

In this section, we express each bisector as an explicit function of the shape $y = f(x)$. Then, the actual bisector is given by the corresponding points (x, y).

Lemma 17 *The bisector $b_{1,2}$ is given by $y = \frac{\sqrt{1-\alpha^2}}{\alpha}(s_x - x)$.*

Proof. We want to compute the coordinates of the points such that the weighted length of paths $\pi_1(s,t)$ and $\pi_2(s,t)$ is the same, i.e., the points (t_x, t_y) such that $\sqrt{(s_x - t_x)^2 + t_y^2} = \alpha(s_x - t_x) + \sqrt{1-\alpha^2}t_y$. Thus:

$$\begin{aligned}
(s_x - t_x)^2 + t_y^2 &= \alpha^2(s_x - t_x)^2 + (1-\alpha^2)t_y^2 \\
&\quad + 2\alpha\sqrt{1-\alpha^2}(s_x - t_x)t_y \\
0 &= (1-\alpha^2)(s_x - t_x)^2 + \alpha^2 t_y^2 \\
&\quad - 2\alpha\sqrt{1-\alpha^2}(s_x - t_x)t_y \\
0 &= \left[\sqrt{1-\alpha^2}(s_x - t_x) - \alpha t_y\right]^2 \\
\alpha t_y &= \sqrt{1-\alpha^2}(s_x - t_x) \\
\Rightarrow t_y &= \frac{\sqrt{1-\alpha^2}}{\alpha}(s_x - t_x). \qquad \square
\end{aligned}$$

Lemma 18 *The bisector $b_{2,3}$ is given by $y = -\frac{\sqrt{1-\alpha^2}}{\alpha}x$.*

Proof. We want to compute the coordinates of the points such that the weighted length of paths $\pi_2(s,t)$ and $\pi_3(s,t)$ is the same, i.e., the points (t_x, t_y) such that $\alpha(s_x - t_x) + \sqrt{1-\alpha^2}t_y = \alpha s_x + \sqrt{t_x^2 + t_y^2}$. Thus:

$$\begin{aligned}
-\alpha t_x + \sqrt{1-\alpha^2}t_y &= \sqrt{t_x^2 + t_y^2} \\
\alpha^2 t_x^2 + (1-\alpha^2)t_y^2 - 2\alpha\sqrt{1-\alpha^2}t_x t_y &= t_x^2 + t_y^2 \\
(1-\alpha^2)t_x^2 + \alpha^2 t_y^2 + 2\alpha\sqrt{1-\alpha^2}t_x t_y &= 0 \\
\left[\sqrt{1-\alpha^2}t_x + \alpha t_y\right]^2 &= 0 \\
\sqrt{1-\alpha^2}t_x &= -\alpha t_y \\
\Rightarrow t_y &= -\frac{\sqrt{1-\alpha^2}}{\alpha}t_x. \qquad \square
\end{aligned}$$

Lemma 19 *The bisector $b_{4,5}$ is given by $y = \frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}}x$.*

Proof. We want to compute the coordinates of the points such that the weighted length of paths $\pi_4(s,t)$ and $\pi_5(s,t)$ is the same, i.e., the points (t_x, t_y) such that $s_x + \sqrt{t_x^2 + t_y^2} = s_x - \sqrt{2-\alpha^2}t_x - \sqrt{\alpha^2-1}t_y$. Thus:

$$\begin{aligned}
&\sqrt{t_x^2 + t_y^2} = -\sqrt{\alpha^2-1}t_y - \sqrt{2-\alpha^2}t_x \\
&\quad t_x^2 + t_y^2 = (\alpha^2-1)t_y^2 + (2-\alpha^2)t_x^2 + 2\sqrt{\alpha^2-1}\sqrt{2-\alpha^2}t_x t_y
\end{aligned}$$

$$
\begin{aligned}
0 &= (2-\alpha^2)t_y^2 - 2\sqrt{\alpha^2-1}\sqrt{2-\alpha^2}t_xt_y + (\alpha^2-1)t_x^2\\
0 &= (\sqrt{2-\alpha^2}t_y - \sqrt{\alpha^2-1}t_x)^2\\
\sqrt{2-\alpha^2}t_y &= \sqrt{\alpha^2-1}t_x\\
\Rightarrow t_y &= \frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}}t_x. \qquad \square
\end{aligned}
$$

Lemma 20 *The bisector $b_{5,6}$ is given by $y = \frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}}x - \sqrt{\alpha^2-1}s_x$.*

Proof. The path $\pi_6(s,t)$ is similar to a path $\pi_5(s,t)$ where the point of entry in R is s. The path $\pi_6(s,t)$ do not have the critical angle property. However, the points on the bisector, still have that critical angle property, since the shortest path from s to them is of both types. Let $(0,b_2)$ be the point where $\pi_5(s,t)$ leaves the square, see Figure 4. Using Equation (5) we obtain the following relation:

$$
\begin{aligned}
\tan\theta_c &= \frac{|s_x|}{|b_2|}\\
\Rightarrow |b_2| &= \frac{|s_x|}{\tan\theta_c} = \sqrt{\alpha^2-1}|s_x| = \sqrt{\alpha^2-1}s_x\\
\Rightarrow b_2 &= -\sqrt{\alpha^2-1}s_x.
\end{aligned}
$$

For $\pi_5(s,t)$, we obtained $b_2 = t_y - \frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}}t_x$ in Lemma 10 (see Equation (8)). We then obtain the equation of the bisector $b_{5,6}$:

$$t_y = b_2 + \frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}}t_x = \frac{\sqrt{\alpha^2-1}}{\sqrt{2-\alpha^2}}t_x - \sqrt{\alpha^2-1}s_x. \qquad \square$$

Lemma 21 *The bisector $b_{7,8}$ is given by $y = -1 - \frac{\sqrt{2-\alpha^2}}{\sqrt{\alpha^2-1}}x$.*

Proof. We want to know the coordinates of the points (t_x,t_y) such that $\sqrt{\alpha^2-1}(s_x+t_x) - \sqrt{2-\alpha^2}(1+t_y) + 1 = \sqrt{\alpha^2-1}s_x + 1 + \sqrt{t_x^2+(t_y+1)^2}$. Thus,

$$
\begin{aligned}
&\sqrt{\alpha^2-1}t_x - \sqrt{2-\alpha^2}(1+t_y) = \sqrt{t_x^2+(1+t_y)^2}\\
&\Rightarrow (\alpha^2-1)t_x^2 + (2-\alpha^2)(1+t_y)^2\\
&\quad -2\sqrt{\alpha^2-1}\sqrt{2-\alpha^2}(1+t_y)t_x = t_x^2 + (1+t_y)^2\\
&\Rightarrow (2-\alpha^2)t_x^2 + (\alpha^2-1)(1+t_y)^2\\
&\quad +2\sqrt{2-\alpha^2}\sqrt{\alpha^2-1}(1+t_y)t_x = 0\\
&\Rightarrow \left[\sqrt{2-\alpha^2}t_x + \sqrt{\alpha^2-1}(1+t_y)\right]^2 = 0\\
&\qquad \Rightarrow -\sqrt{\alpha^2-1}(1+t_y) = \sqrt{2-\alpha^2}t_x\\
&\qquad \Rightarrow -1 - \frac{\sqrt{2-\alpha^2}}{\sqrt{\alpha^2-1}}t_x = t_y. \qquad \square
\end{aligned}
$$

Lemma 22 *The bisector $b_{10,11}$ is given by $y = -\sqrt{\alpha^2-1}(s_x-x)$.*

Proof. We want the curve defining the bisector between the region containing the points $t=(t_x,t_y)$ such that the shortest path from $s=(s_x,0)$ to t is $\pi_{10}(s,t)$, and the region containing the points $t=(t_x,t_y)$ such that the shortest path from $s=(s_x,0)$ to t is $\pi_{11}(s,t)$. Thus,

$$\alpha\sqrt{(s_x-t_x)^2+t_y^2} = s_x - t_x - \sqrt{\alpha^2-1}t_y$$

$$
\begin{aligned}
\Rightarrow \alpha^2[(s_x-t_x)^2+t_y^2] &= (s_x-t_x)^2 + (\alpha^2-1)t_y^2\\
&\quad - 2(s_x-t_x)\sqrt{\alpha^2-1}t_y\\
\Rightarrow 0 &= \alpha^2(s_x-t_x)^2 - (s_x-t_x)^2 + \alpha^2t_y^2\\
&\quad - \alpha^2t_y^2 + t_y^2 + 2(s_x-t_x)\sqrt{\alpha^2-1}t_y\\
&= (\alpha^2-1)(s_x-t_x)^2 + t_y^2\\
&\quad + 2(s_x-t_x)^2\sqrt{\alpha^2-1}t_y\\
&= (t_y + \sqrt{\alpha^2-1}(s_x-t_x))^2\\
\Rightarrow t_y &= -\sqrt{\alpha^2-1}(s_x-t_x). \qquad \square
\end{aligned}
$$

Lemma 23 *The bisector $b_{11,12}$ bisector is given by $y = -\frac{(s_x+x)+2\alpha\sqrt{s_x x}}{\sqrt{\alpha^2-1}}$.*

Proof. We want the curve defining the bisector between the region containing the points $t=(t_x,t_y)$ such that the shortest path from $s=(s_x,0)$ to t is $\pi_{11}(s,t)$, and the region containing the points $t=(t_x,t_y)$ such that the shortest path from $s=(s_x,0)$ to t is $\pi_{12}(s,t)$. Thus,

$$
\begin{aligned}
&\sqrt{\alpha^2-1}(s_x+t_x) - t_y = \alpha\sqrt{(s_x-t_x)^2+t_y^2}\\
\Rightarrow &\left[\sqrt{\alpha^2-1}(s_x+t_x) - t_y\right]^2 = \alpha^2((s_x-t_x)^2+t_y^2).
\end{aligned}
$$

By expanding the square we find

$$
\begin{aligned}
0 &= (\alpha^2-1)t_y^2 + 2\sqrt{\alpha^2-1}(s_x+t_x)t_y + \alpha^2(s_x-t_x)^2\\
&\quad - (\alpha^2-1)(s_x+t_x)^2\\
&= (\alpha^2-1)t_y^2 + 2\sqrt{\alpha^2-1}(s_x+t_x)t_y + \alpha^2s_x^2 + \alpha^2t_x^2\\
&\quad - 2\alpha^2s_xt_x - \alpha^2s_x^2 - \alpha^2t_x^2 - 2\alpha^2s_xt_x + (s_x+t_x)^2\\
&= (\alpha^2-1)t_y^2 + 2\sqrt{\alpha^2-1}(s_x+t_x)t_y - 4\alpha^2s_xt_x\\
&\quad + (s_x+t_x)^2.
\end{aligned}
$$

From which we obtain that

$$
\begin{aligned}
t_y &= \frac{-2\sqrt{\alpha^2-1}(s_x+t_x)}{2(\alpha^2-1)}\\
&\quad \pm \frac{\sqrt{4(\alpha^2-1)((s_x+t_x)^2 - [-4\alpha^2s_xt_x + (s_x+t_x)^2])}}{2(\alpha^2-1)}\\
&= \frac{-\sqrt{\alpha^2-1}(s_x+t_x) \pm \sqrt{(\alpha^2-1)4\alpha^2s_xt_x}}{\alpha^2-1}\\
&= \frac{-(s_x+t_x) \pm 2\alpha\sqrt{s_xt_x}}{\sqrt{\alpha^2-1}}.
\end{aligned}
$$

Let $(0,b_1)$ and $(0,b_2)$ be, respectively, the points where $\pi_{12}(s,t)$ leaves and enters for the second time the region R. We know that $b_1 > b_2$. Thus, using Lemma 16, we know that $\pi_{12}(s,t)$ exists if $t_y < -\frac{s_x+t_x}{\sqrt{\alpha^2-1}}$. Hence, the bisector is given by the curve $t_y = \frac{-(s_x+t_x)-2\alpha\sqrt{s_xt_x}}{\sqrt{\alpha^2-1}}$. $\square$