CLEAR

Unit-Distance Graphs

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Puzzles
Puzzles

Unit Distance Graph:

- all edges \((p, q)\) have \(d(p, q) = 1\)
- all non-edges \((p, q)\) have \(d(p, q) \neq 1\)
Puzzles

Clear Unit Distance Graph:

• all edges \((p, q)\) have \(d(p, q) = 1\)
• all non-edges \((p, q)\) have \(d(p, q) \in [\varepsilon, 1 - \varepsilon] \cup [1 + \varepsilon, \infty)\)
Connect-the-dots
Unite-the-dots

Connect all pairs $p, q$ with $d(p, q) = 1$
Unite-the-dots

Connect all pairs $p, q$ with $d(p, q) = 1$
Unite-the-dots

Which points are at distance 1 from \( p \)?
Which points are at distance 1 from $p$?

It should be clear which pairs to connect for all pairs $p, q$, we require

$$d(p, q) \in [\varepsilon, 1 - \varepsilon] \cup [1] \cup [1 + \varepsilon, \infty)$$
Unite-the-dots

Which points are at distance 1 from $p$?

It should be clear which pairs to connect the points should be the vertices of a clear unit distance graph.
Properties of Clear UD Graphs

Density

#points in region of constant diameter?

(Geometric) Diameter

Size of the paper required to draw the graph?

Number of Crossings
Properties of Clear UD Graphs

Density

Given a \((1, \varepsilon)\)-graph.

\(#\) points in region of constant diameter?

Upperbound: \(O(1/\varepsilon^2)\)
Properties of Clear UD Graphs

Density
Given a \((1, \varepsilon)\)-graph.

\# points in region of constant diameter?

Upperbound: \(O(1/\varepsilon^2)\)

Witness: \(\Omega(1/\varepsilon^2)\)
Properties of Clear UD Graphs

Density

Given a connected \((1, \varepsilon)\)-graph.

\# points in region of constant diameter?

Upperbound: \(O(1/\varepsilon^2)\)

Witness: \(\Omega(1/\varepsilon^2)\)
Properties of Clear UD Graphs

Density
Given a \((1, \varepsilon)\)-path.

\#points in region of constant diameter?

Upperbound: \(O(1/\varepsilon^2)\)

Witness: \(\Omega(1/\sqrt{\varepsilon})\)
Properties of Clear UD Graphs

Diameter

Given a connected $(1, \varepsilon)$-graph.

What is the (geometric) diameter?

Upperbound: $O(n)$ trivial
Properties of Clear UD Graphs

Diameter
Given a connected $(1, \varepsilon)$-graph, with $0 < \varepsilon \leq \sqrt{3} - 1$. What is the (geometric) diameter?

Upperbound: $O(n)$ trivial

Witness: $\Omega(n)$
Properties of Clear UD Graphs

Diameter
- Given a connected \((1, \varepsilon)\)-graph.
- What is the (geometric) diameter?

Upperbound: \(O(n)\)  trivial
Lowerbound: \(\Omega(\sqrt{n\varepsilon})\)
Witness: \(\Omega(\sqrt{n\varepsilon})\)
Properties of Clear UD Graphs

Diameter
Witness: $\Omega(\sqrt{n\varepsilon})$
Unite-the-dots

Input:

Output:
Unite-the-dots

Input:

Output:
Unite-the-dots

$p$ and $q$ u-model $C$ iff

- $d(p, q) = 1$
- $\|C\| \leq 1 + \delta$
- $C$ inside both unit discs centered at $p$ and $q$
$p_1, \ldots, p_k$ u-model $C$ iff

- $p_1$ and $p_k$ are the endpoints of $C$
- $p_i$ and $p_{i+1}$ u-model $C(p_i, p_{i+1})$
- All other $p_i, p_j$ have $d(p_i, p_j) \neq 1$
\(p_1, \ldots, p_k\) u-model \(C\) iff 

- \(p_1\) and \(p_k\) are the endpoints of \(C\)
- \(p_i\) and \(p_{i+1}\) u-model \(C(p_i, p_{i+1})\)
- All other \(p_i, p_j\) have \(d(p_i, p_j) \in [\varepsilon, 1 - \varepsilon] \cup [1 + \varepsilon, \infty)\)
Unite-the-dots

\[ p_1, \ldots, p_k \text{ u-model } C \]

we allow one piece to be not u-modelled.
$P$ u-models $C_1, \ldots, C_h$ iff

- All $C_i$’s are u-modelled by $P' \subseteq P$
- All other $p$ and $q$ have $d(p, q) \in [\varepsilon, 1 - \varepsilon] \cup [1 + \varepsilon, \infty)$
Algorithm
Pre-drawn piece at the end
Algorithm

Pre-drawn piece at the end

$P_i$

$Q_i$
Algorithm

Pre-drawn piece at the end

Each curve $C_i$ 2 choices $P_i$ or $Q_i$

$\iff$

$x_i = \text{TRUE}$ and $x_i = \text{FALSE}$

Build a 2-SAT formula:

if $Q_i \cup P_j$ not a $(1, \varepsilon)$-point set then add $x_i \lor \overline{x_j}$
Algorithm

Pre-drawn piece at the end

$P_i$

$Q_i$

Running time: $O(n^2)$
Algorithm

Pre-drawn piece at the end

Running time: $O\left(\frac{n}{\varepsilon^2 \log n}\right)$
Algorithm

Running time: $O(n/\varepsilon^2 \log n)$

Depends on density bound for $(1, \varepsilon)$-paths

Future Work: Improve to ...?
Algorithm

Pre-drawn piece at the end

$P_i$

$Q_i$

Running time: $O(n/\varepsilon^2 \log n)$

Interior piece: Similar approach
Algorithm

Pre-drawn piece at the end

$P_i$

$Q_i$

Running time: $O(n/\varepsilon^2 \log n)$

Interior piece:
Minimize piece length:
$> 1$ piece per curve:

Similar approach

NP hard
NP hard
Thank you!
Thank you!