Clear Unit-Distance Graphs*

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Abstract

We introduce a variation of unit-distance graphs which we call *clear unit-distance graphs*. They require the pairwise distances of the representing points to be either exactly 1 or not close to 1. We discuss properties and applications of clear unit-distance graphs.

1 Introduction

Unit-distance graphs are embedded graphs, usually in $\mathbb{R}^2$, with the property that there is an edge between two vertices if and only if their distance is exactly 1. Unit-distance graphs can be traced back to Erdös [3], who asked the now-famous open question of how many edges a unit-distance graph with n vertices can have. Since then, many deep observations about the class of unit-distance graphs have been made [1, 2, 5].

Unit-distance graphs appear in many applications, but we focus here on the role they play in certain games and puzzles. Figure 1 shows a maze puzzle realized by a set of a holes in a metal plate, and a movable ring that has a small part of the perimeter missing. The missing part allows the ring to navigate from hole to hole, but only between holes at unit distance.

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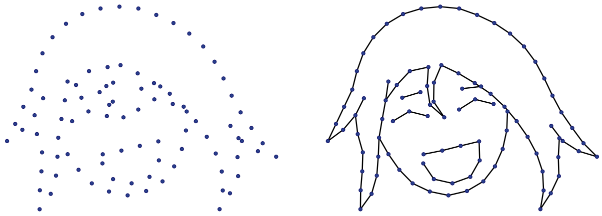
Figure 1: A maze represented as the graph of a clear unit distance point set. The goal is to get the ring through the two holes connected by a line segment (the text reads "GOAL").



(a) (b)

Figure 2: (a) A set of points, forming a unite-the-dots puzzle. (b) The drawing induced by the points as a clear unit distance graph, with $\varepsilon = 0.1$.

As another example, we propose a new type of drawing puzzle. The puzzle *connect-the-dots* is a well-known activity for young children, where the goal is to create a drawing by connecting a series of numbered dots in the given order. The resulting curve usually completes a simple picture. However, connecting the dots by means of numbered dots suffers from some profound disadvantages. The puzzle only allows for a single polyline, which decreases the possible complexity of the drawing. Usually this is circumvented by pre-drawing some parts of the drawing. In addition, the printed numbers do not improve the visual appearance of the completed drawing. Finally, the use of numbers is not particularly challenging. We therefore propose an adapted version of this puzzle which is both more challenging and allows not just polyline drawings, without the use of numbering. In this puzzle, which we call *unite-the-dots*, two dots should be connected by a line segment if the distance between these two points is equal to a pre-defined distance. Without loss of generality we assume this distance to be 1, hence the drawing resulting from a unite-the-dots puzzle corresponds to a unit-distance graph. An example is shown in Figure 2.

Clear unit-distance graphs. In practice, it is hard to distinguish points at unit distance, and points at almost unit distance. This motivates studying a new class of graphs, the *clear unit-distance graphs*, in which points are either at unit distance, or a distance significantly different from 1. Clear unit-distance graphs can be seen as a variation on the unit-distance graph that is useful in practical situations like the ones just described. We will model the minimum required deviation from the unit distance by an additive parameter ε. Moreover, in the applications mentioned above, it is also undesirable if pairs of points are too close. For unite-the-dots, the disk-shaped drawings of points must be disjoint, and for the mechanical mazes the holes should not merge into bigger holes. For simplicity we use the same parameter ε to specify how far two distinct points should be apart. Hence, for a given value ε, the allowed distances between points lie in the range $[\varepsilon, 1-\varepsilon] \cup [1] \cup [1+\varepsilon, \infty)$.

Results and Organization. Section 2 formally introduces clear unit-distance graphs and Section 3 investigates several properties of these graphs. Section 4 describes the unite-the-dots problem in more detail and briefly discusses how these properties may be used to obtain efficient algorithms for automatically generating unite-the-dots puzzles.

2 Definitions

An *ε-distinguishable unit-distance point set* is a set of points in the plane with the property that every pair (p, q) of points has a distance $d(p, q)$ that is either exactly 1, or at least ε and at most $1-\varepsilon$, or at least $1+\varepsilon$.[1] An ε-distinguishable unit-distance point set induces an *ε-distinguishable unit-distance drawing*, by connecting all points at unit distance with an edge. A graph G is an *ε-distinguishable unit-distance graph* if it has an *ε-distinguishable unit-distance drawing* whose points correspond to the vertices of G. For brevity we will write $(1, \varepsilon)$-point set, $(1, \varepsilon)$-graph, and $(1, \varepsilon)$-drawing.

A *clear unit-distance graph* is an ε-distinguishable unit-distance graph for some constant $\varepsilon > 0$.

3 Properties of clear unit-distance graphs

In this section, we investigate some properties of $(1, \varepsilon)$-point sets and graphs. We assume $0 < \varepsilon < 1$ is a given constant, and write $\gamma = 1/\varepsilon$.

3.1 Density bounds

Observation 1 *Let G be an $(1, \varepsilon)$-graph, and $R \subset \mathbb{R}^2$ a region of constant diameter. There can be at most $O(\gamma^2)$ vertices of any $(1, \varepsilon)$-drawing of G in R.*

[1]While the extra condition that inter-point distance must be at least ε is natural from the point of view of the applications, the conceptually cleaner definition which only requires distances to be either 1 or at least ε different from 1 may also be of theoretical interest. We argue that the extra condition does not influence the results too much, since any two points at distance less than ε must have exactly the same neighbors in the (relaxed) $(1, \varepsilon)$-graph. We defer a more thorough discussion of this issue to the full paper.

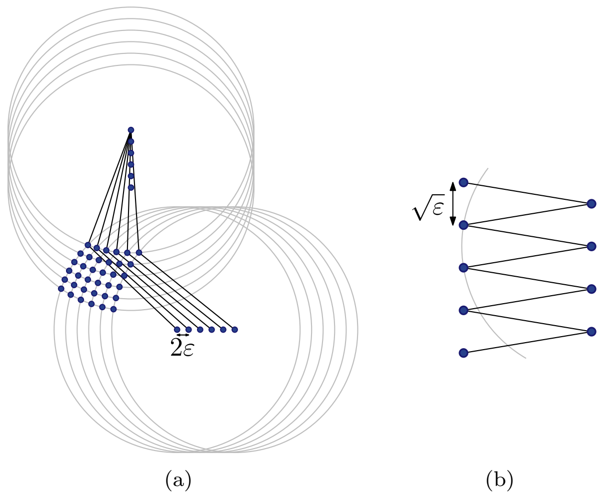


Figure 3: (a) A connected graph may have $\Omega(\gamma^2)$ vertices in a constant-diameter area. (Not all edges are shown to avoid visual clutter.) (b) A zig-zag path must have distance $\sqrt{\varepsilon}$ between points to ensure a distance of at least ε from a point to the unit circle centered at the points on the other side.

This upper bound follows directly from the requirement that inter-point distances are at least ε. More interestingly, the bound can actually be achieved, even if the graph is required to be connected:

Lemma 2 *There exist a connected $(1, \varepsilon)$-graph G, a $(1, \varepsilon)$-drawing D of G and a region of constant diameter R, such that there are $\Omega(\gamma^2)$ vertices of D in R.*

Proof. Note that placing points on the vertices of a regular ε-spaced $\gamma/4$ by $\gamma/4$ grid results in a valid $(1, \varepsilon)$-drawing, since all inter-point distances are at least ε and clearly smaller than $1-\varepsilon$. To make the graph connected, we slightly modify the grid and place the points on the intersections of two sets of $\gamma/4$ circles, whose centers lie 2ε-spaced on a horizontal and a vertical line, see Figure 3(a). □

The construction above relies on high-degree vertices to work. More interesting is the question of how dense a connected $(1, \varepsilon)$-graph with maximum vertex degree d can be. Already for the case $d = 2$ (i.e., for paths) this appears to be a challenging question. We provide a lower bound of $\Omega(\sqrt{\gamma})$:

Lemma 3 *There exist a connected $(1, \varepsilon)$-graph G of maximum degree 2, a $(1, \varepsilon)$-drawing D of G and a region of constant diameter R, such that there are $\Omega(\sqrt{\gamma})$ vertices of D in R.*

Proof. We take two sets of $\sqrt{\gamma}/2$ points, each lying $\sqrt{\varepsilon}$-spaced on a vertical line, such that the first two points on the first line both lie at distance 1 to the first point on the second line. Then the induced

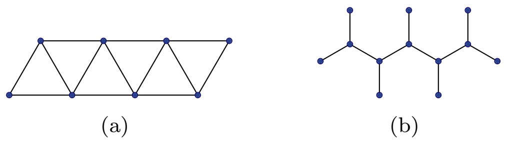


Figure 4: (a) A fixed triangle strip forces a large diameter. (b) For $\varepsilon = \sqrt{3}-1$, a claw graph has a fixed drawing, which can be replicated to get a linear diameter.

unit-distance graph is a path. Furthermore, the distance between any pair of points not at distance 1 is at distance more than $1+\varepsilon$ if they lie on different lines or less than $1/2$ if they lie on the same line, see Figure 3(b). □

3.2 Crossing number bounds

The number of crossings in the example of Figure 3(a) is $\Theta(\gamma^4)$, showing that a constant-diameter region can have as many crossings.

Lemma 4 *Any $(1,\varepsilon)$-drawing has $O(\gamma^{16/3})$ crossings in a constant-diameter region.*

Proof. Only $O(\gamma^2)$ points can be within unit distance of a constant-diameter region because these points must lie in a (slightly larger) constant-diameter region themselves. These points realize at most $O(\gamma^{8/3})$ unit distances which are the edges (using the upper bound for the unit-distance problem by Spencer, Szemerédi and Trotter [4]). □

3.3 Diameter bounds

We now proceed to bound the (geometric) diameter of $(1,\varepsilon)$-drawings. Clearly, since all connected point pairs are at unit distance, no drawing of a connected $(1,\varepsilon)$-graph can have diameter larger than n. We show that $(1,\varepsilon)$-graphs exist whose drawings necessarily need this diameter:

Lemma 5 *A connected $(1,\varepsilon)$-graph G of maximum degree 4 exists such that any $(1,\varepsilon)$-drawing of G has diameter $\Omega(n)$, for any $0 < \varepsilon \leq \sqrt{3}-1$.*

Proof. A rigid strip of triangles is a clear unit distance graph. See Figure 4(a). □

For some specific values of ε, we can show that even trees may require a linear diameter:

Lemma 6 *A $(1,\varepsilon)$-tree G of maximum degree 3 exists such that any $(1,\varepsilon)$-drawing of G has diameter $\Omega(n)$, for $\varepsilon = \sqrt{3}-1$.*

Proof. Let G be a caterpillar graph where all internal nodes have degree 3 (i.e., a path turned into a tree by adding a leaf to every internal path node). Now, the three incident edges of any internal node must make $120°$ angles with each other; otherwise, two of them would be at a distance less than $\sqrt{3} = 1+\varepsilon$ to each other (note that they are not connected in G so they cannot be at distance 1, and they also cannot be at a distance bigger than ε and smaller than $1-\varepsilon$ since $\varepsilon > 1/2$). It remains to argue that the embedding has a purely zigzagging backbone. But this is obvious, since any deviation would place six points in regular hexagonal position, creating a cycle. □

Clearly, as a direct consequence of Observation 1, a $(1,\varepsilon)$-drawing must have diameter at least $\Omega(\sqrt{n\varepsilon})$. Conversely, we show that there are graphs for which every drawing has this diameter:

Lemma 7 *A connected $(1,\varepsilon)$-graph G exists such that any $(1,\varepsilon)$-drawing of G has diameter $O(\sqrt{n\varepsilon})$.*

Proof. We construct a graph G consisting of $O(\varepsilon^2 n)$ copies of the construction in Figure 3(a), linked to a $c\sqrt{n\varepsilon}$ by $c\sqrt{n\varepsilon}$ grid. Clearly, the grid ensures that any drawing has diameter $O(\sqrt{n\varepsilon})$, and by choosing c sufficiently large we make sure that if the grid is drawn regularly, there is enough room in its faces for the $O(\gamma^2)$ points without interfering with the grid points themselves. □

4 Unite-the-dots

Unite-the-dots puzzles are a variation of connect-the-dots (a.k.a. follow-the-dots), where a set of numbered points is given, and a polyline must be drawn that connects them in the right order. Unite-the-dots does not use numbers to annotate points. Instead, two points are connected if and only if they are at exactly the right (unit) distance. Unit-distance drawings are the output for a given set of points, and clear unit-distance point sets are suitable as the input for unite-the-dots puzzles. The puzzle can be solved with the help of a small coin or short stick.

In this section we study the problem of converting a line drawing into a clear unit-distance point set whose clear unit-distance drawing resembles the line drawing. Let C be a curve between two points p, q at unit distance. We say that the line segment $\overline{pq}$ *u-models* C with respect to a parameter $\delta \geq 0$ if the length of C is at most $1+\delta$, and C is fully inside the intersection of the radius-1 disks centered at p and q. By extension, we also say that the points p, q u-model C.

More generally, let C be any curve. Denote the subcurve between any two points $p, q \in C$ by $C(p,q)$. Let $p_1, \ldots, p_k$ be a set of k points on C and ordered along it. Then we say that $p_1, \ldots, p_k$ u-model C if (i) p_1 and p_k coincide with the two endpoints of C,

(ii) points p_i, p_{i+1} are at unit distance for all $1 \leq i \leq k-1$, (iii) points p_i, p_{i+1} u-model $C(p_i, p_{i+1})$ for all $1 \leq i \leq k-1$, and (iv) no other pair of points is at unit distance. To be suitable as a $(1, \varepsilon)$-point set, we need to strengthen the last condition: (iv) every other pair of points is at distance $\geq \varepsilon$ and $\leq 1-\varepsilon$ or $\geq 1+\varepsilon$.

Even more generally, let $\mathcal{C} = \{C_1, \ldots, C_h\}$ be a collection of curves. A $(1, \varepsilon)$-set of points P u-models $\mathcal{C}$ if and only if P is a $(1, \varepsilon)$-point set, for every curve, a subset of the points in P u-models it, and no pair of points of P lies at distance 1 unless they u-model a piece of some curve in $\mathcal{C}$. Intuitively, this means that the corresponding $(1, \varepsilon)$-drawing resembles $\mathcal{C}$. Furthermore, we require that P be minimal: no subset of P should also u-model $\mathcal{C}$. This condition ensures that P has no isolated points in its $(1, \varepsilon)$-drawing.

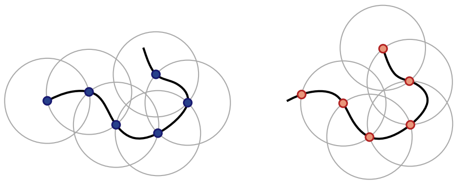


Figure 5: A curve with points chosen from the one end or the other end.

Most curves, even most line segments, are not u-modeled by any point set. Since we must choose one point of P at the endpoint of the curve, and the next point must be at unit distance, it would be a coincidence if a point (the last point) coincides with the other endpoint. For example, only integer-length line segments can be u-modeled. To overcome this caveat we will allow one piece of each curve to be not u-modeled. This may be an end piece or an interior piece, but we would like it to be short. One could for instance start at one end of C, choose a point in P, and then incrementally choose the next point where the curve leaves the unit disk around the previously chosen point, until the remaining part of C lies fully inside the last disk, see Figure 5. We would then have to check whether the chosen points $p_1, \ldots, p_k$ u-model $C(p_1, p_k)$ and form a $(1, \varepsilon)$-point set. Similarly, we could start at the other end, or start at both ends and leave a part in the middle.

Given a drawing, represented by a set of curves, we wish to compute the dots that make a unite-the-dots puzzle, along with any curve pieces that are not u-modeled by the dots. We observe by the u-modeling definition with parameter δ that the number of points in P is always linear in the total length of the curves in the input.

Suppose we are given a collection $\mathcal{C}$ of h curves and parameters ε and δ, we can decide whether a $(1, \varepsilon)$-point set exists whose $(1, \varepsilon)$-drawing resembles $\mathcal{C}$ with the exception of at most one short ending piece per curve (which would be pre-drawn). This is done as follows. For each curve C_i, generate the two sets of points P_i and Q_i as in Figure 5. They are associated with the TRUE and FALSE states of a variable x_i. By examining P_i and Q_i separately we can decide if x_i can be TRUE or FALSE at all. By taking pairs of points of different curves, say a point of Q_i and a point of P_j, we test their distance to decide if they can be together in a $(1, \varepsilon)$-point set. If not, we make a clause $(x_i \vee \overline{x}_j)$. This approach allows us to solve the problem using 2-SAT in $O(n^2)$ time, where n is the total length of all curves (and also the number of points in a unite-the-dots puzzle, if one exists).

Using packing and algorithmic ideas we can improve the bound to $O(n \log n)$. The dependency on ε is quartic, which can be derived from our results in Section 3. We can use the same approach when we allow to pre-draw at most one short interior curve piece. Minimizing the total length of the pieces that must be pre-drawn is NP-hard. We show these results in the full paper.

5 Conclusions

We introduced clear unit-distance graphs and drawings as a way to model situations where it is important to clearly distinguish unit-distance point pairs from other point pairs. We made several observations about the properties of clear unit-distance graphs. We expect that many of our bounds can be improved. It would be of particular interest to improve the density upper bound for paths: there is a substantial gap between the $\Omega(\sqrt{\gamma})$ lower bound and the $O(\gamma^2)$ upper bound, and an improved upper bound would immediately imply a better ε-dependency for our unite-the-dots algorithm.

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