Computing the Expected Area of an Induced Triangle

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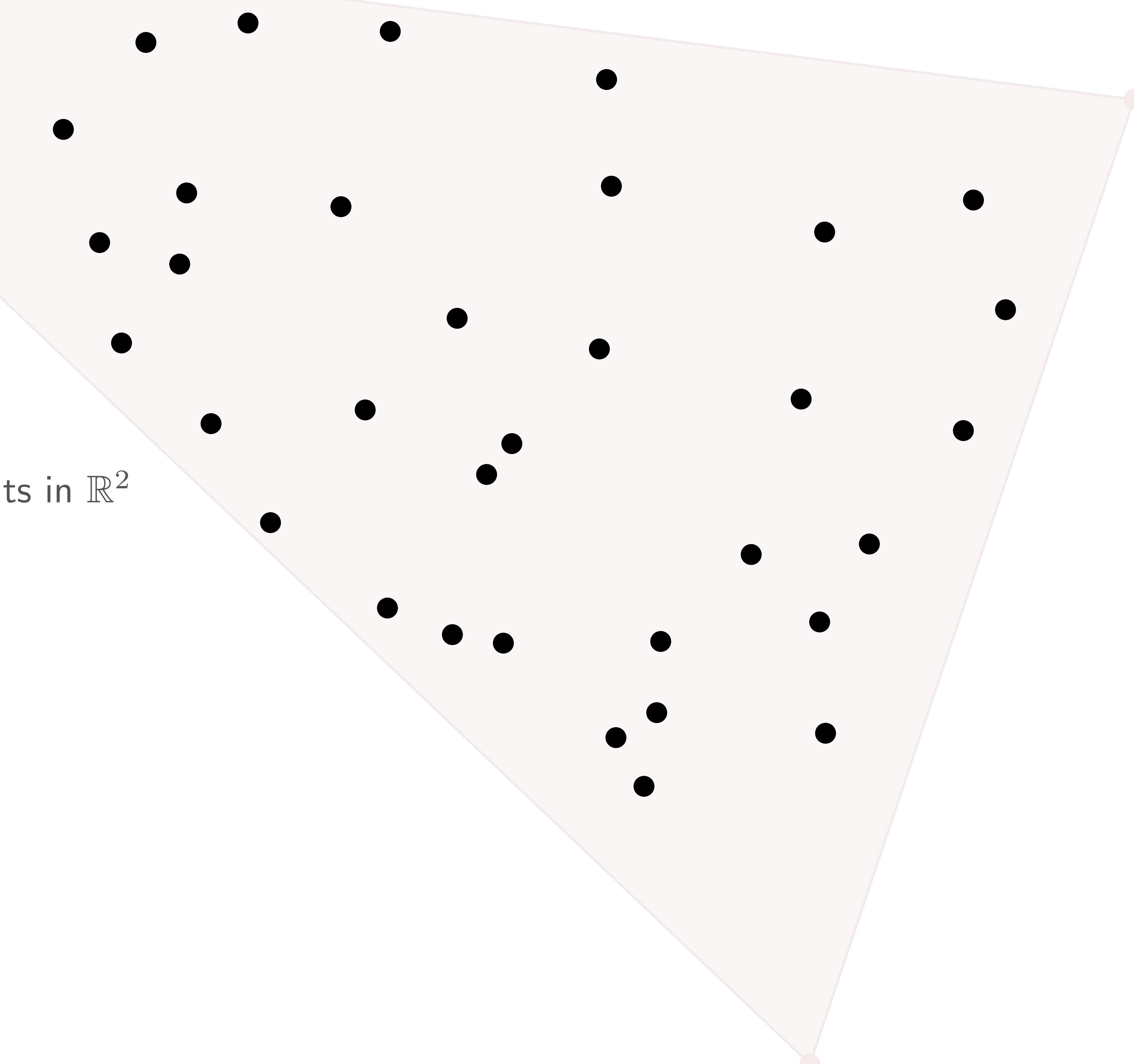
Problem Statement



Given: a set P of n points in $\mathbb{R}^2$

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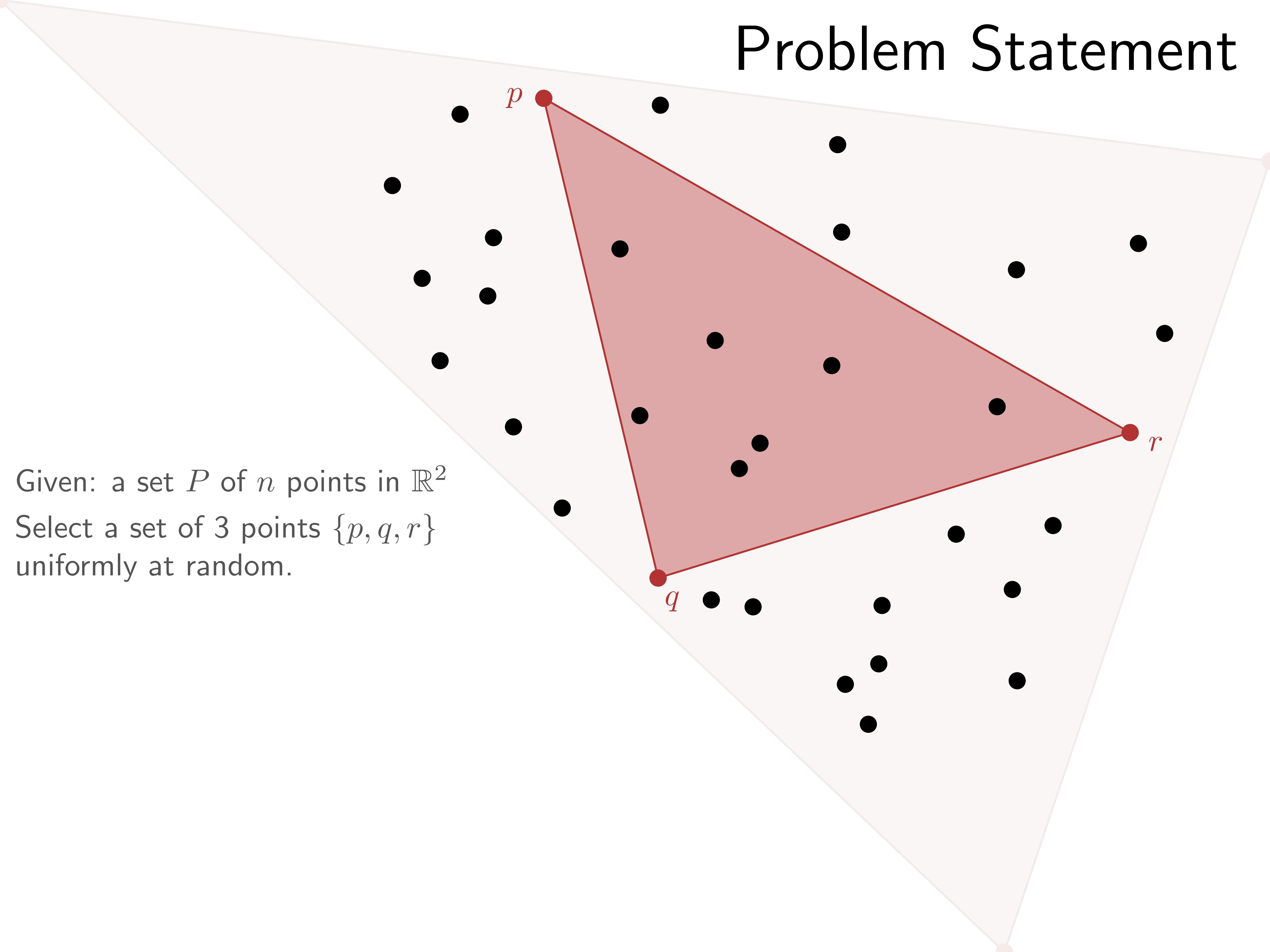

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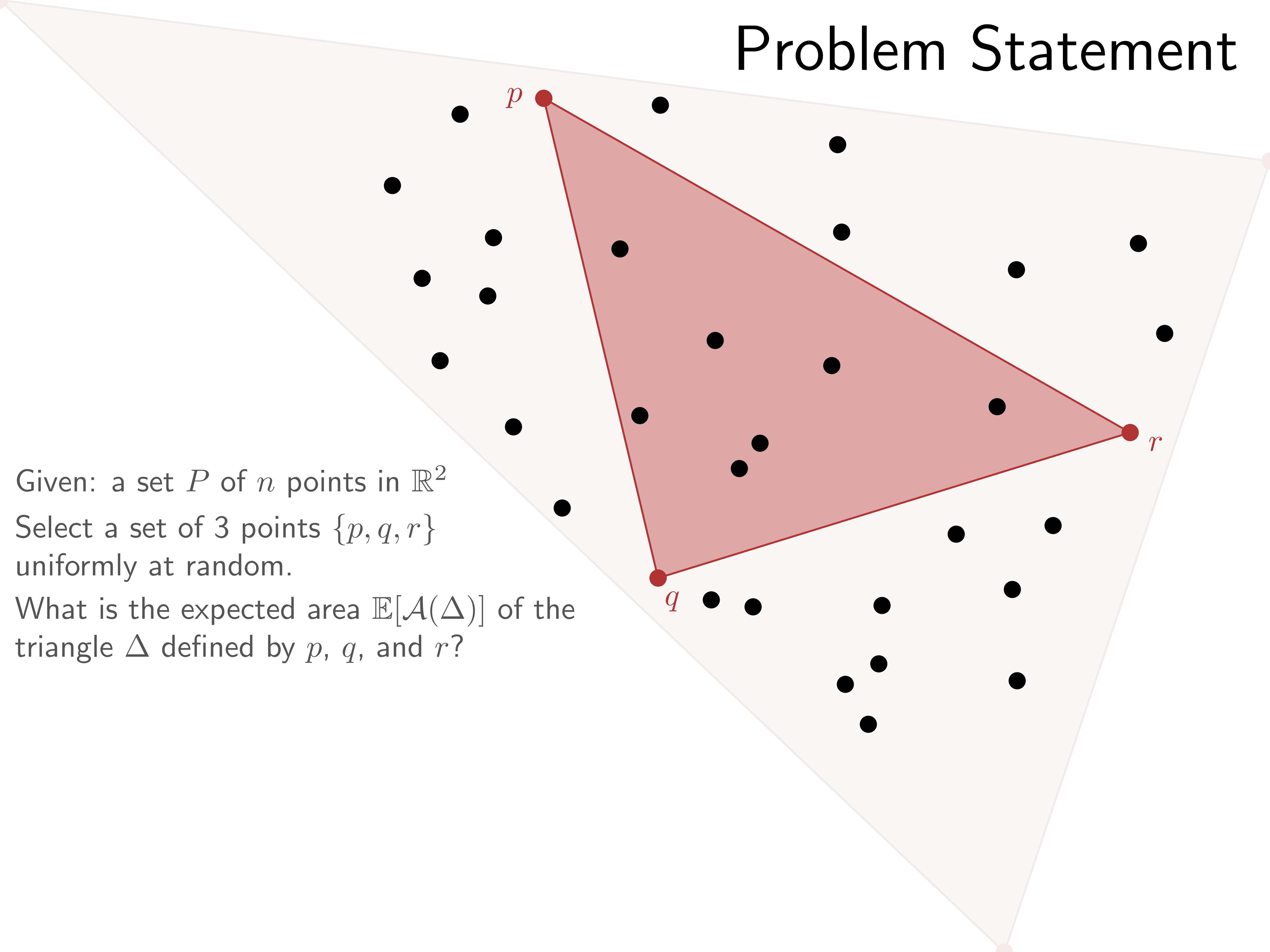

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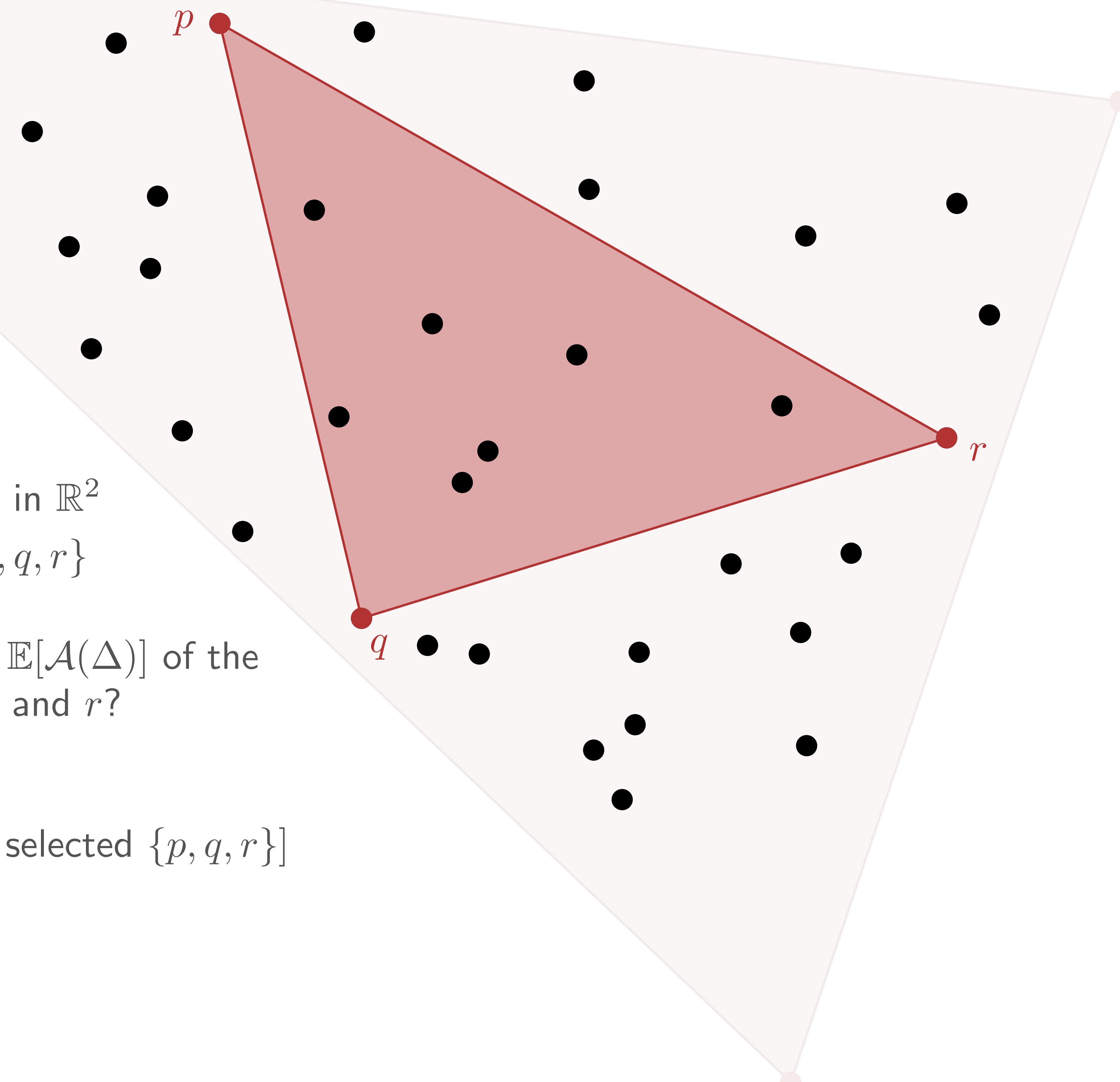

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$$\sum_{\{p,q,r\} \subseteq P} \mathcal{A}(\Delta)\mathbb{P}[\text{we selected } \{p, q, r\}]$$

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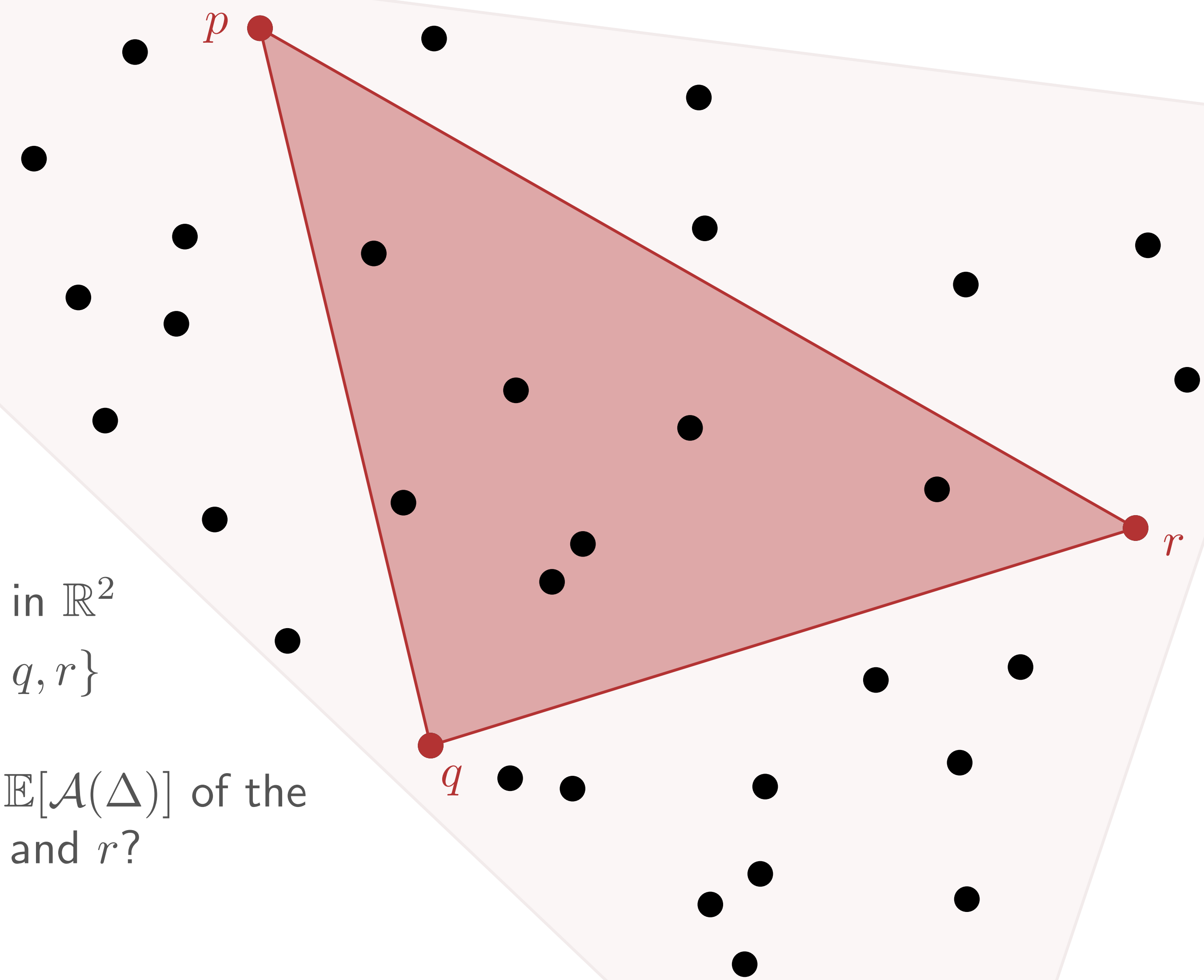

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Naive: $O(n^3)$

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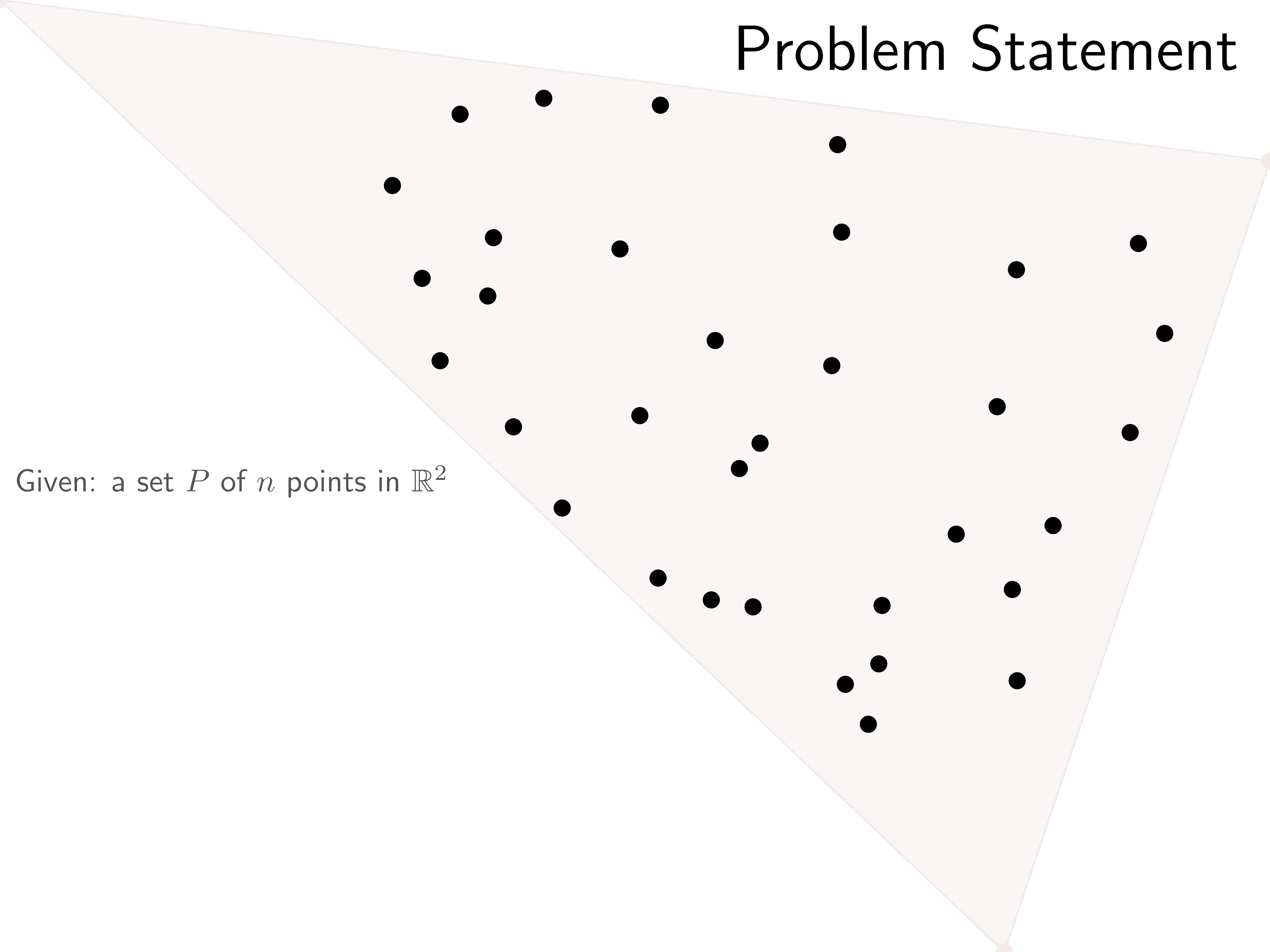

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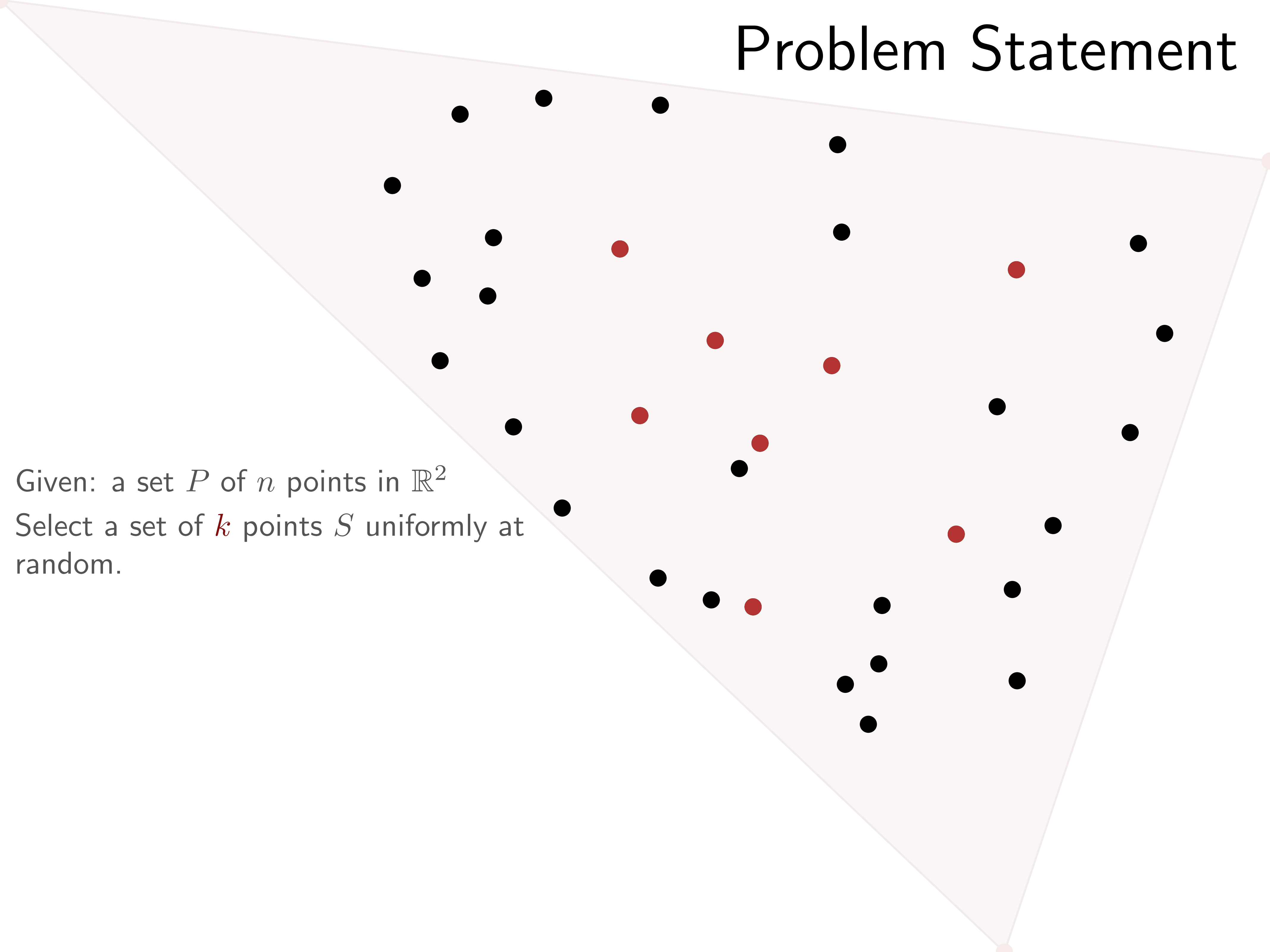

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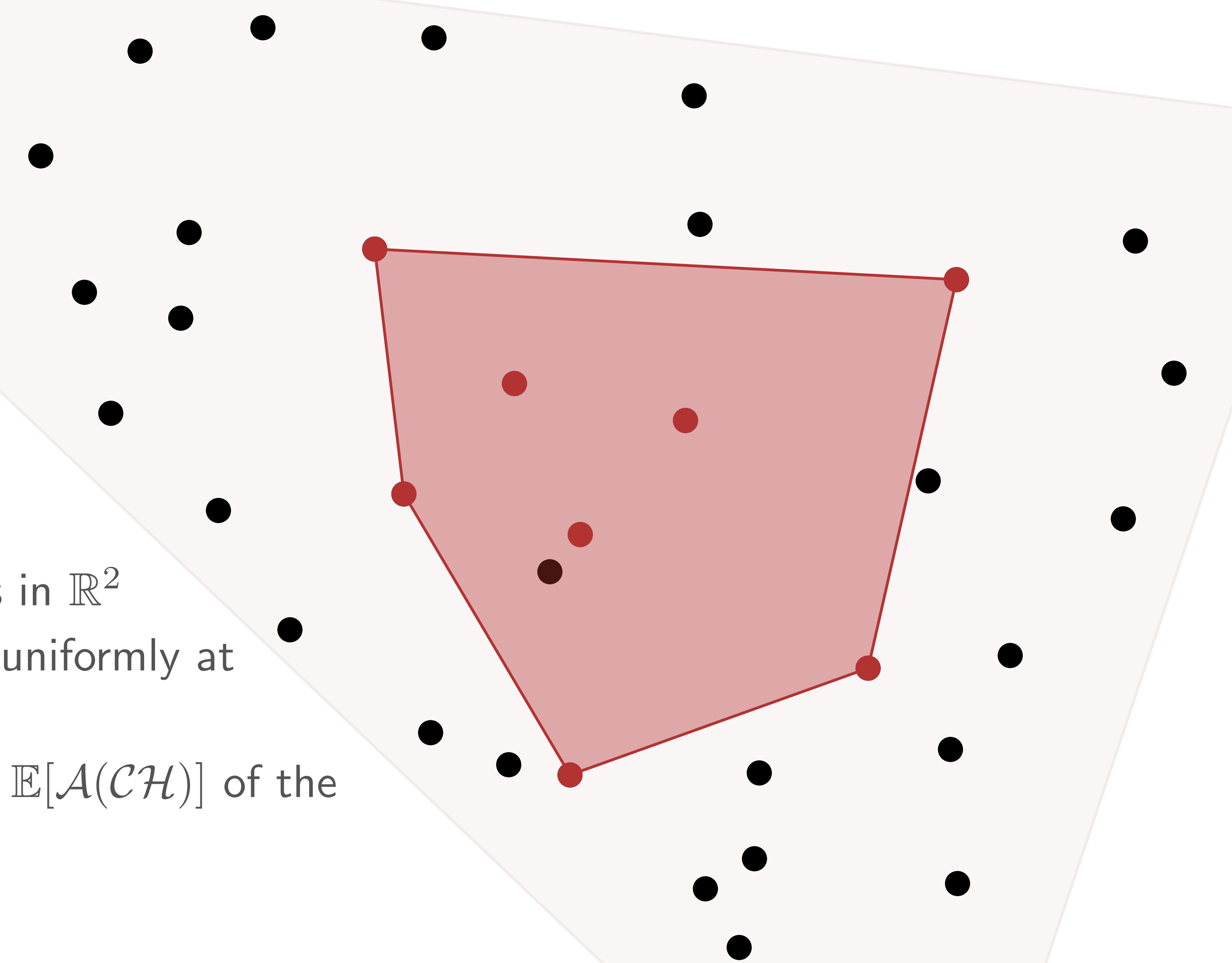

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What is the expected area $\mathbb{E}[\mathcal{A}(\mathcal{CH})]$ of the convex hull $\mathcal{CH}$ of S

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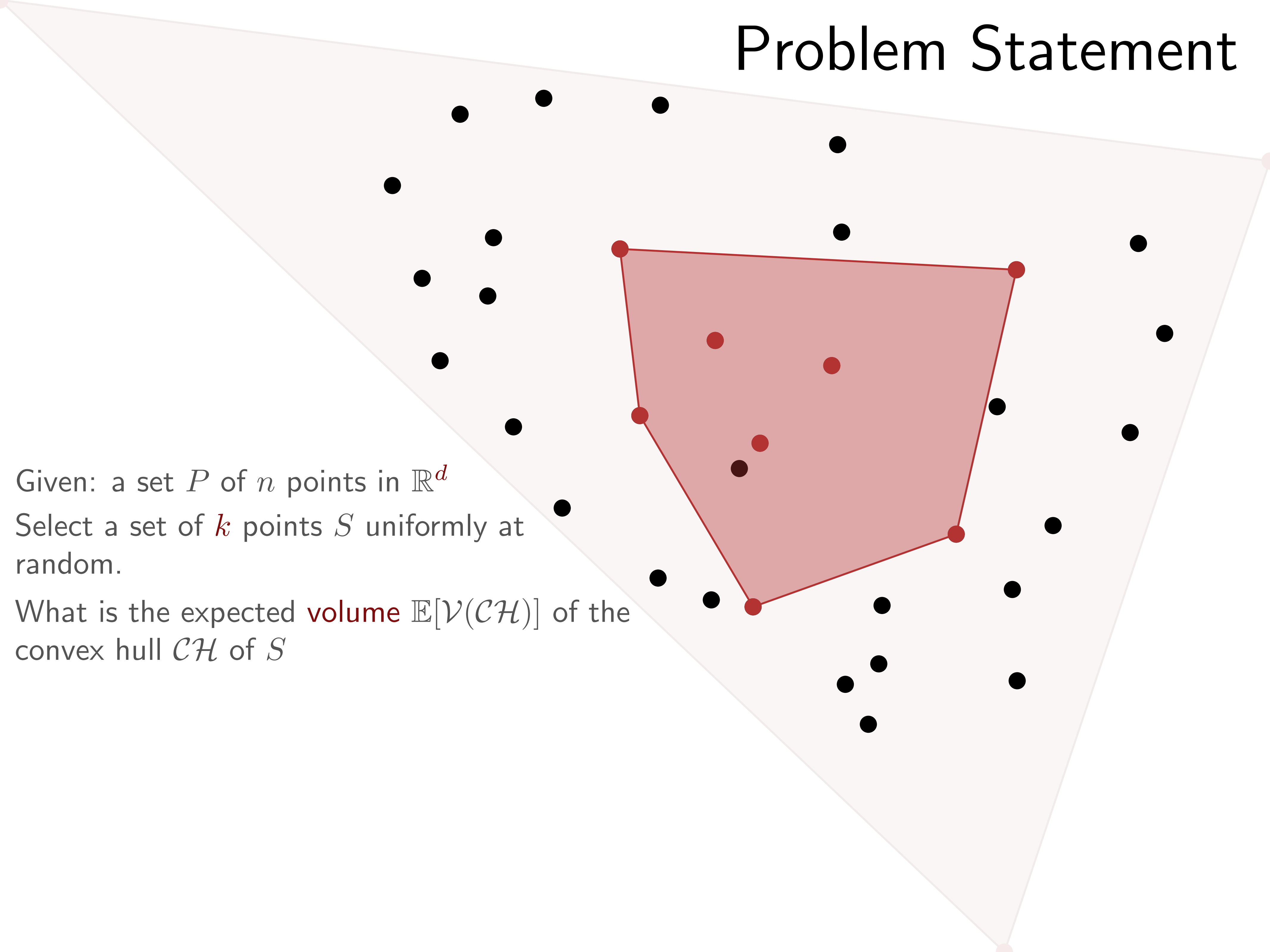

Given: a set P of n points in $\mathbb{R}^d$

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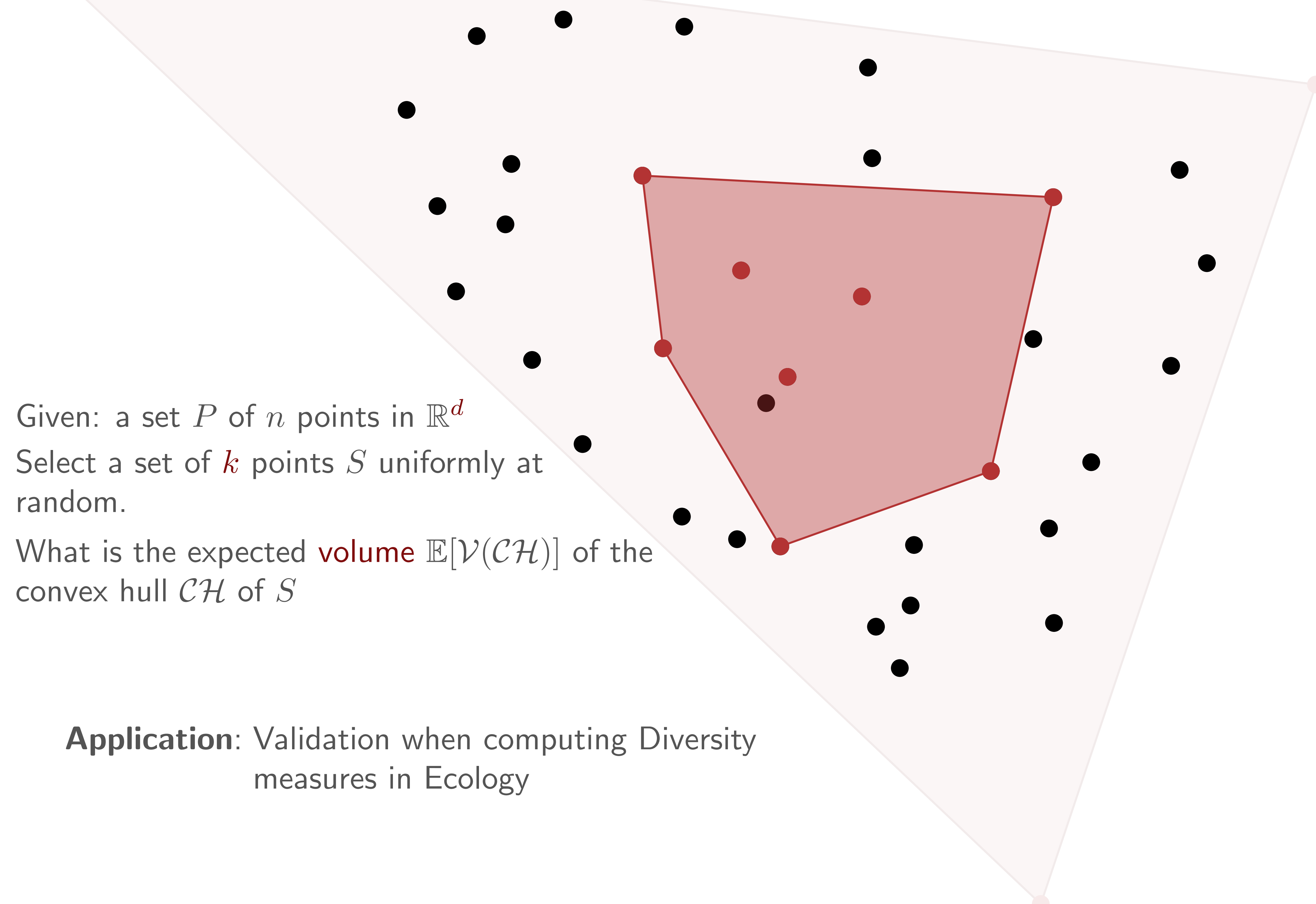

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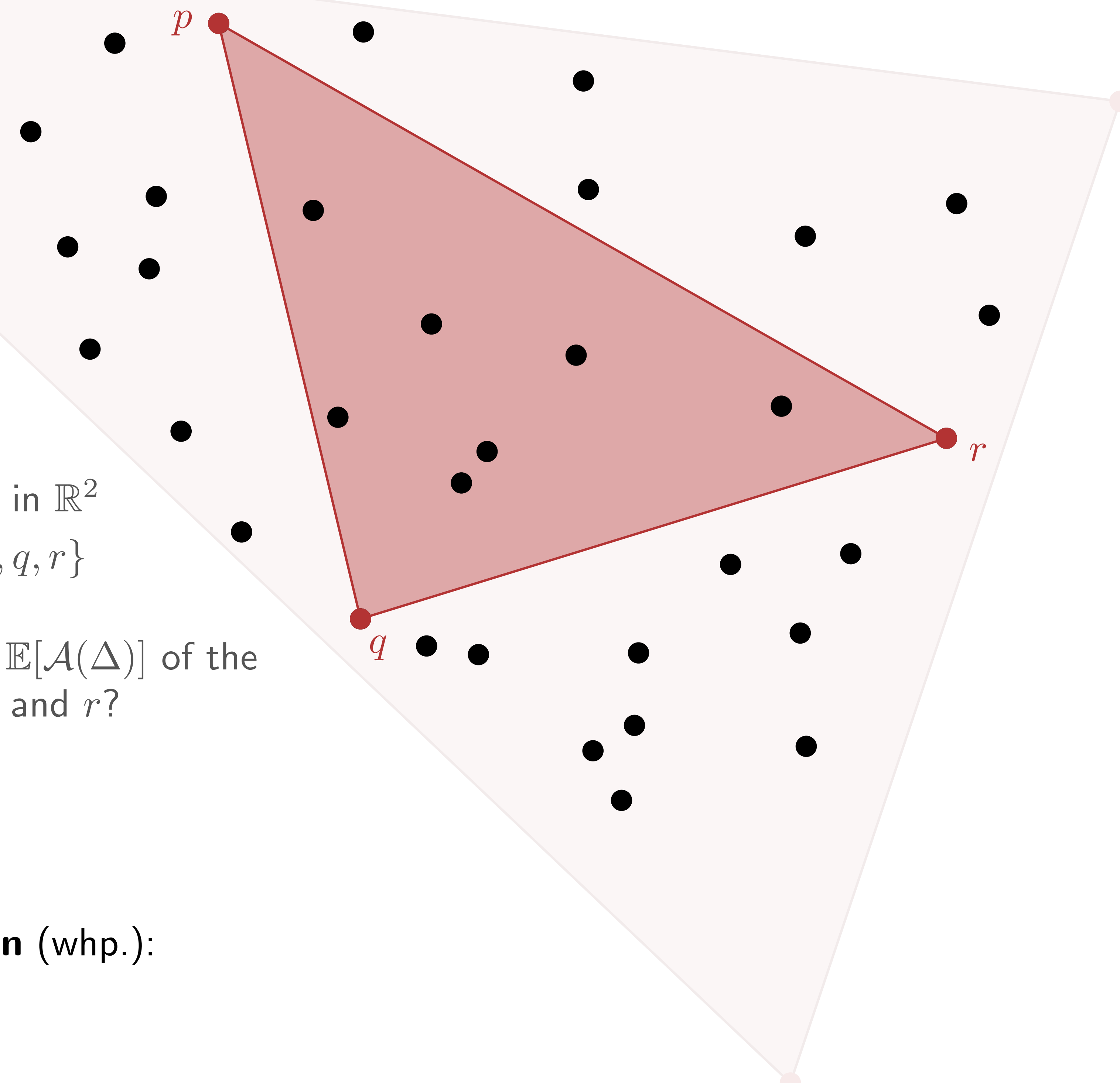
Application: Validation when computing Diversity measures in Ecology

Results



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Exact: $O(n^2 \log n)$

$(1 \pm \varepsilon)$-**Approximation** (whp.):

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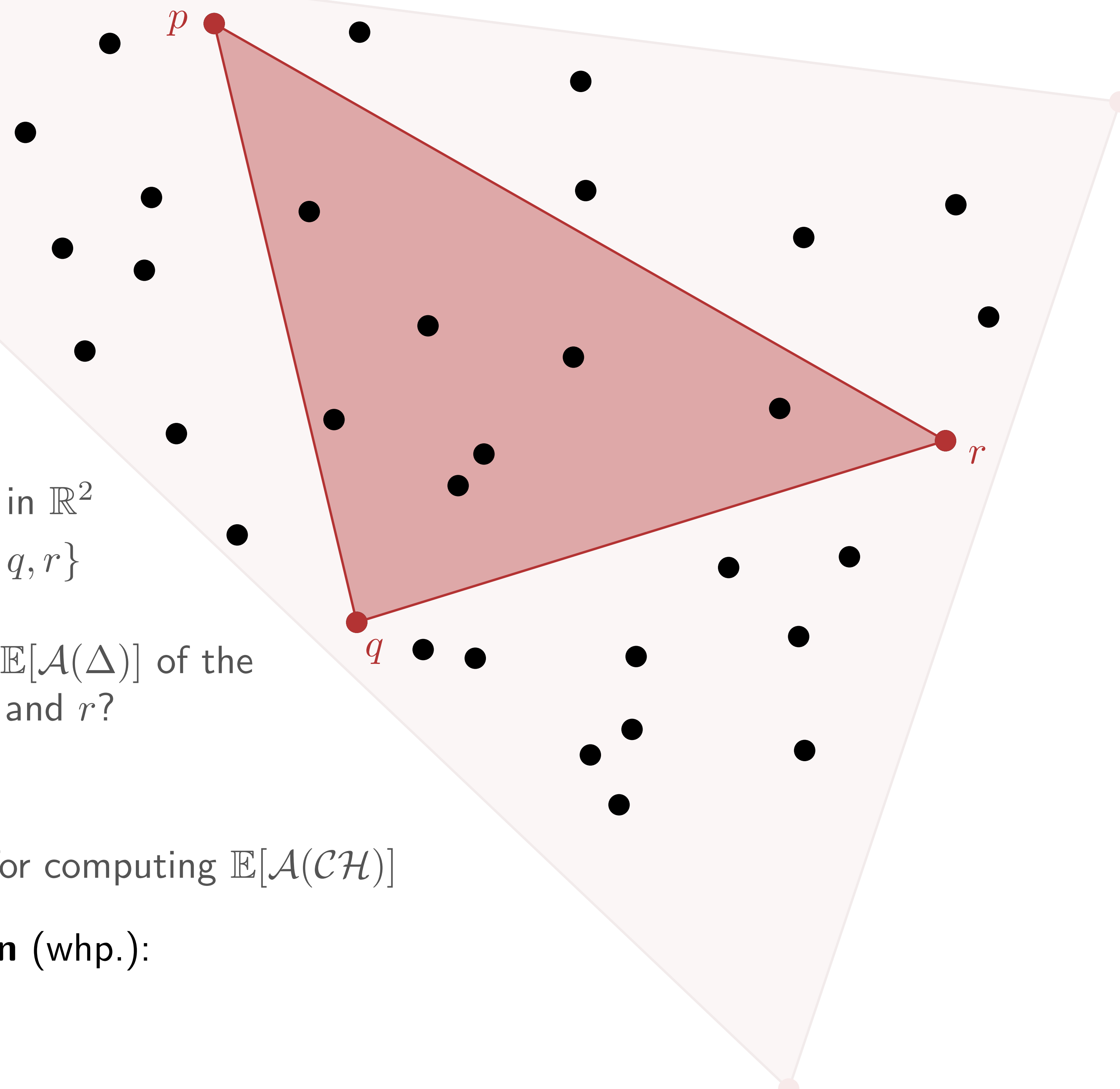

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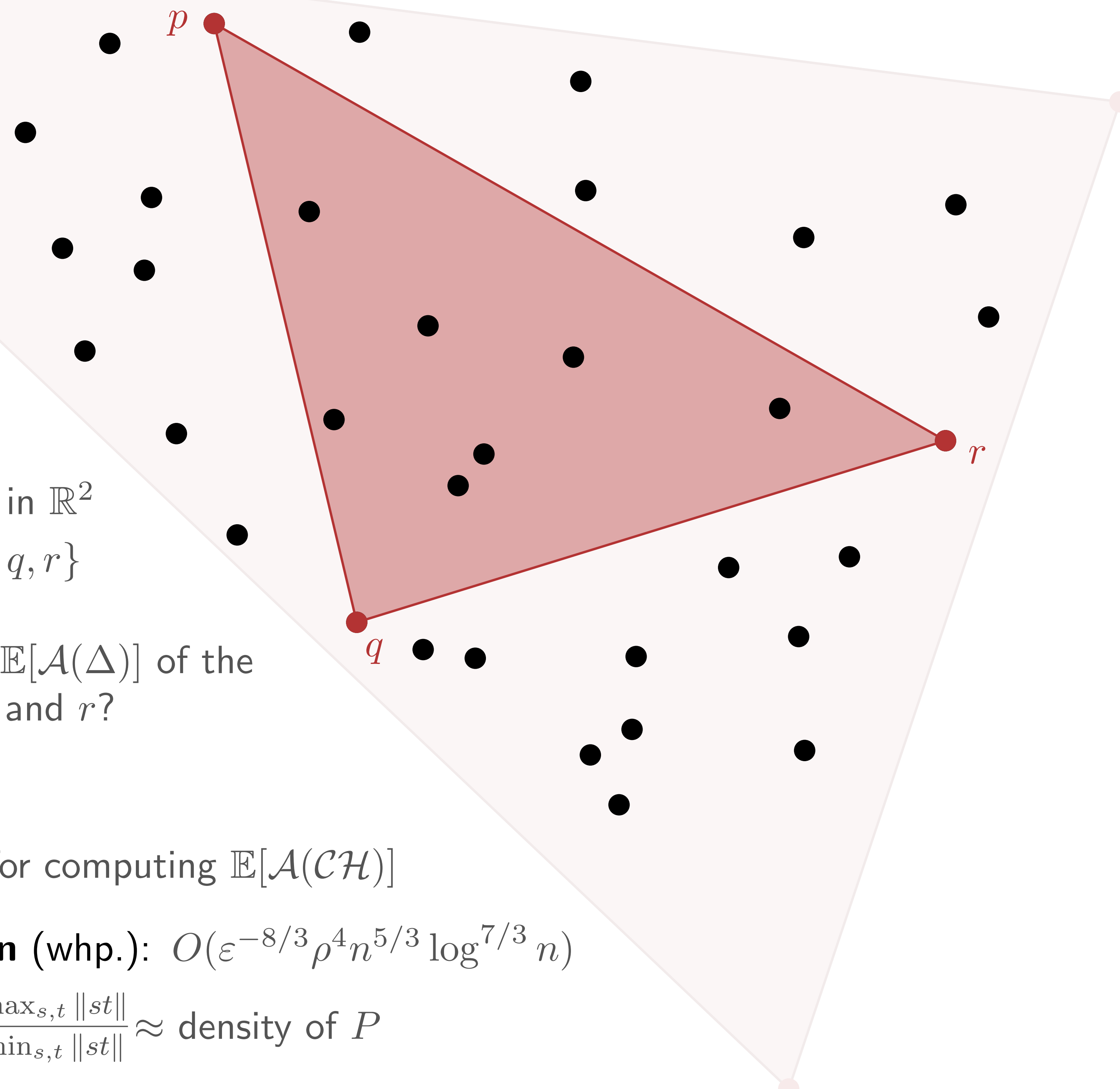

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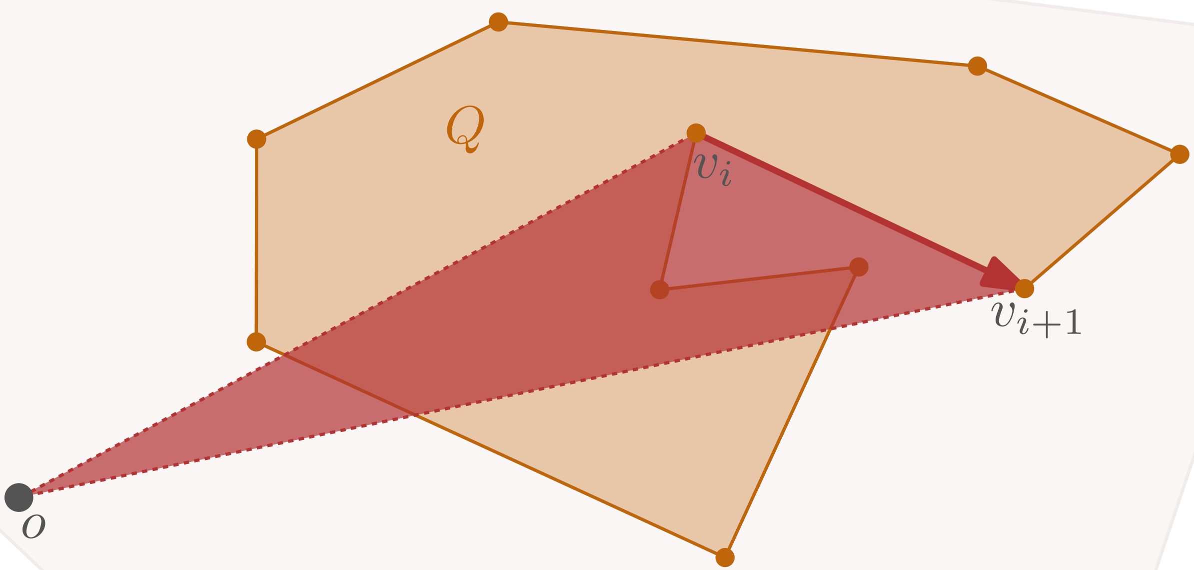
Also works for computing $\mathbb{E}[\mathcal{A}(\mathcal{CH})]$

$(1 \pm \varepsilon)$-**Approximation** (whp.): $O(\varepsilon^{-8/3} \rho^4 n^{5/3} \log^{7/3} n)$

$\rho = \frac{\max_{s,t} \|st\|}{\min_{s,t} \|st\|} \approx$ density of P

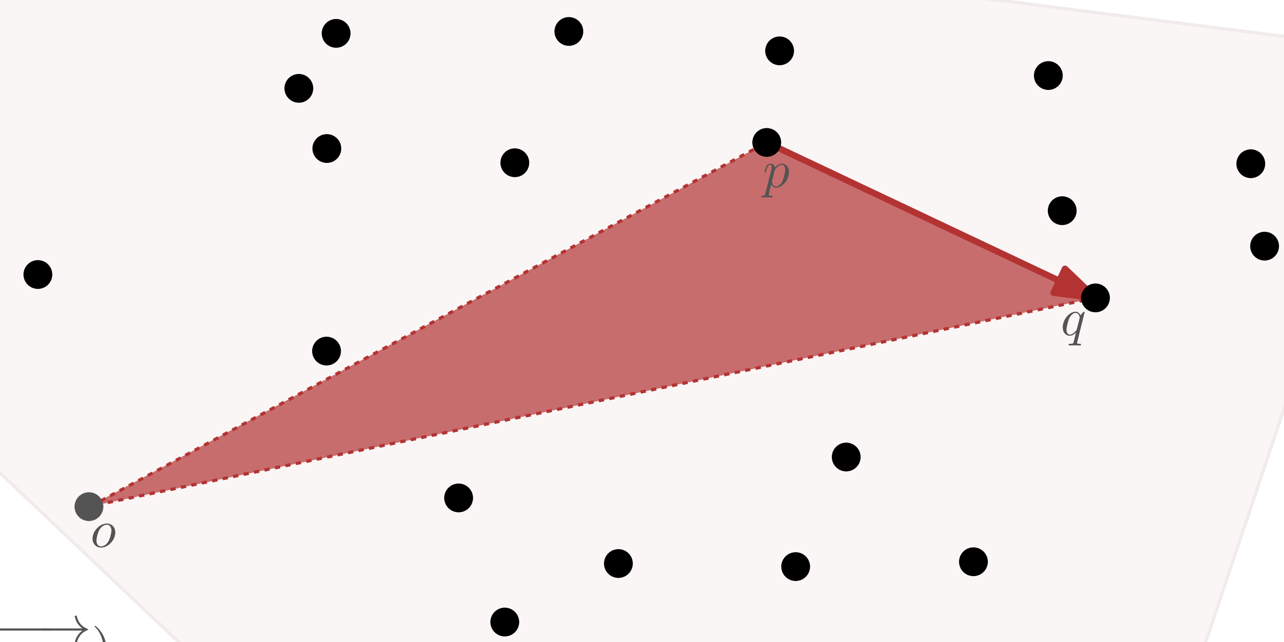
Exact Algorithm



Shoelace formula

$$\mathcal{A}(Q) = \frac{1}{2}\sum_{i=0}^{n} \mathcal{A}'(\overrightarrow{v_i v_{i+1 \mod n}})$$

Exact Algorithm



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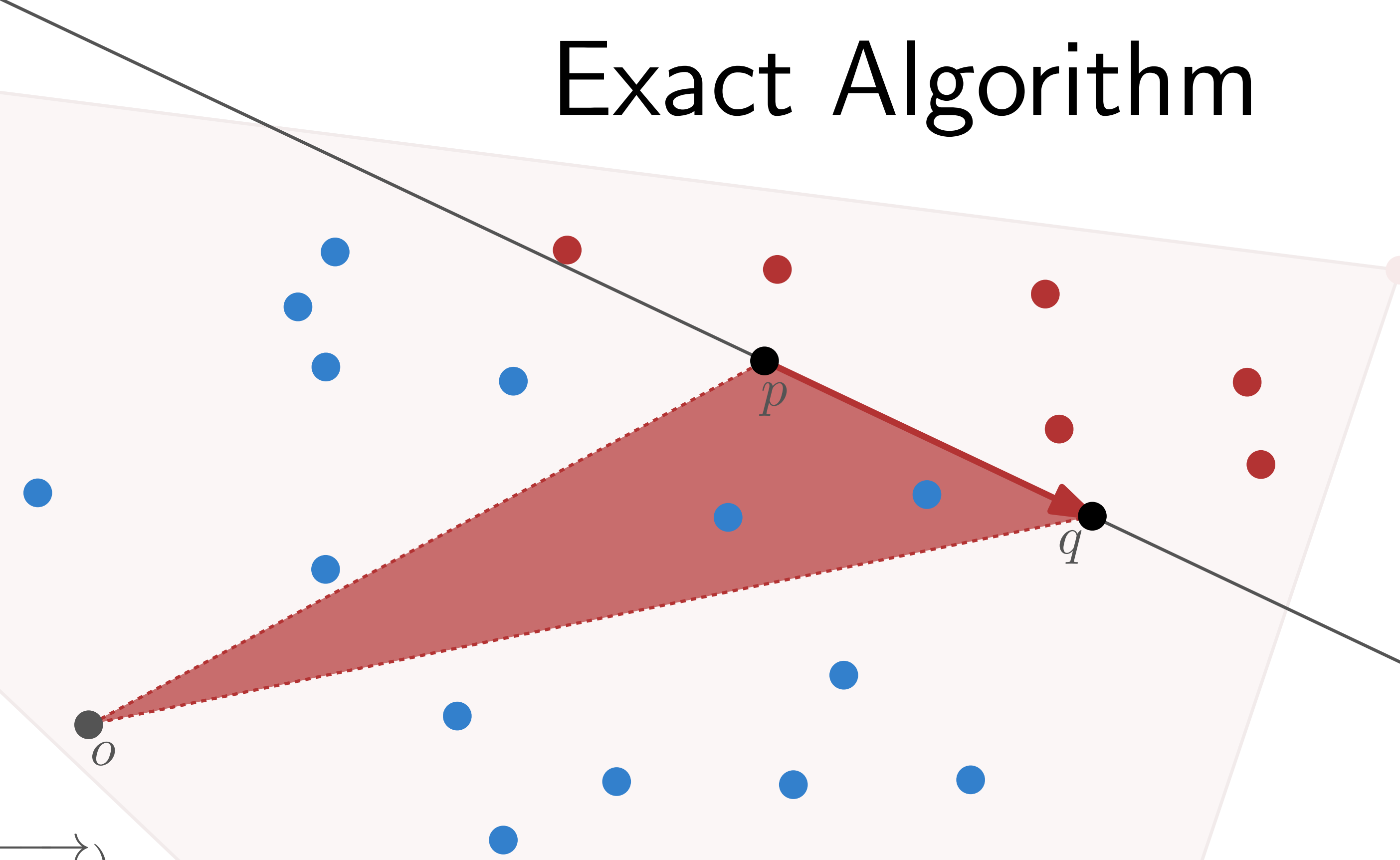

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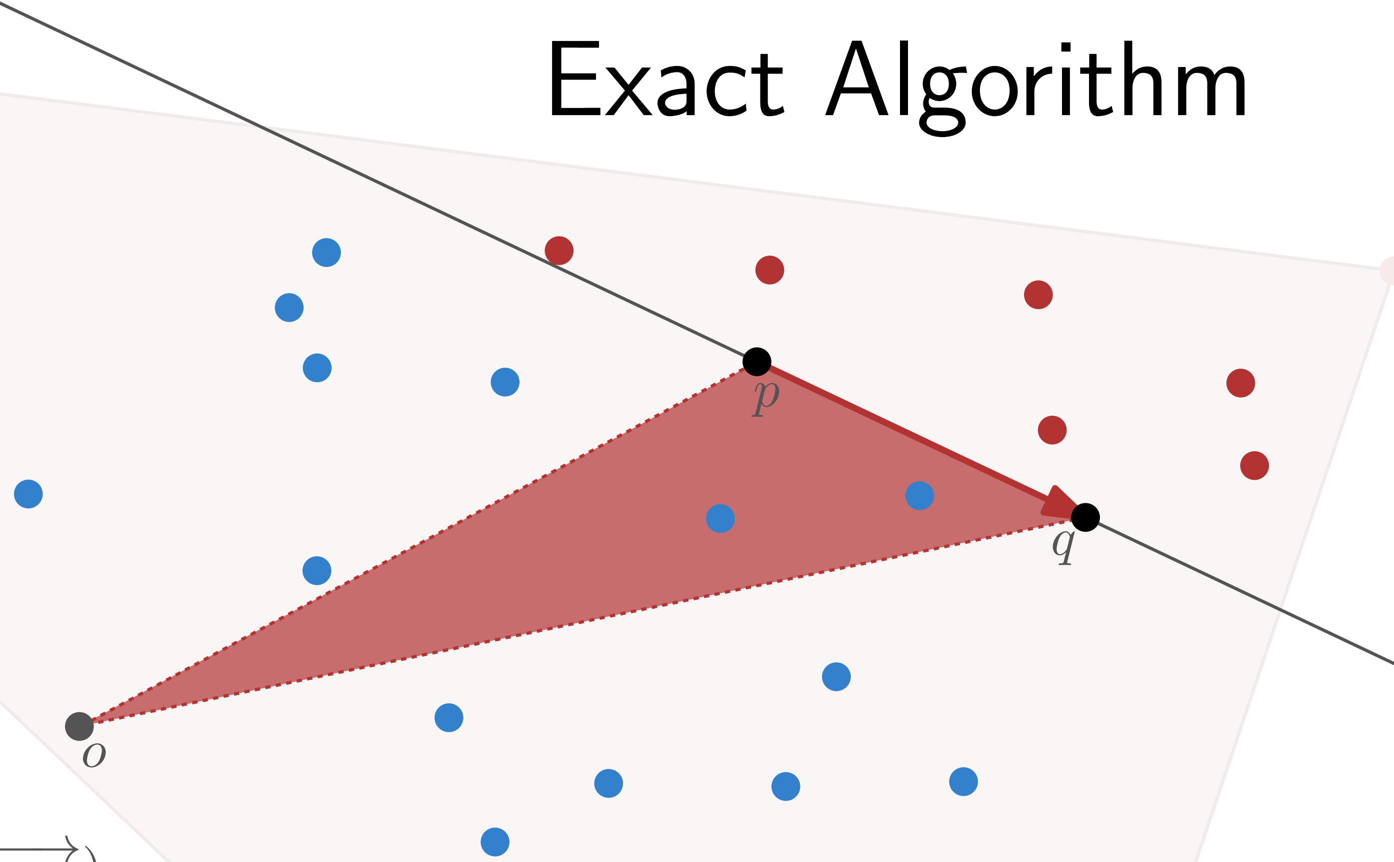

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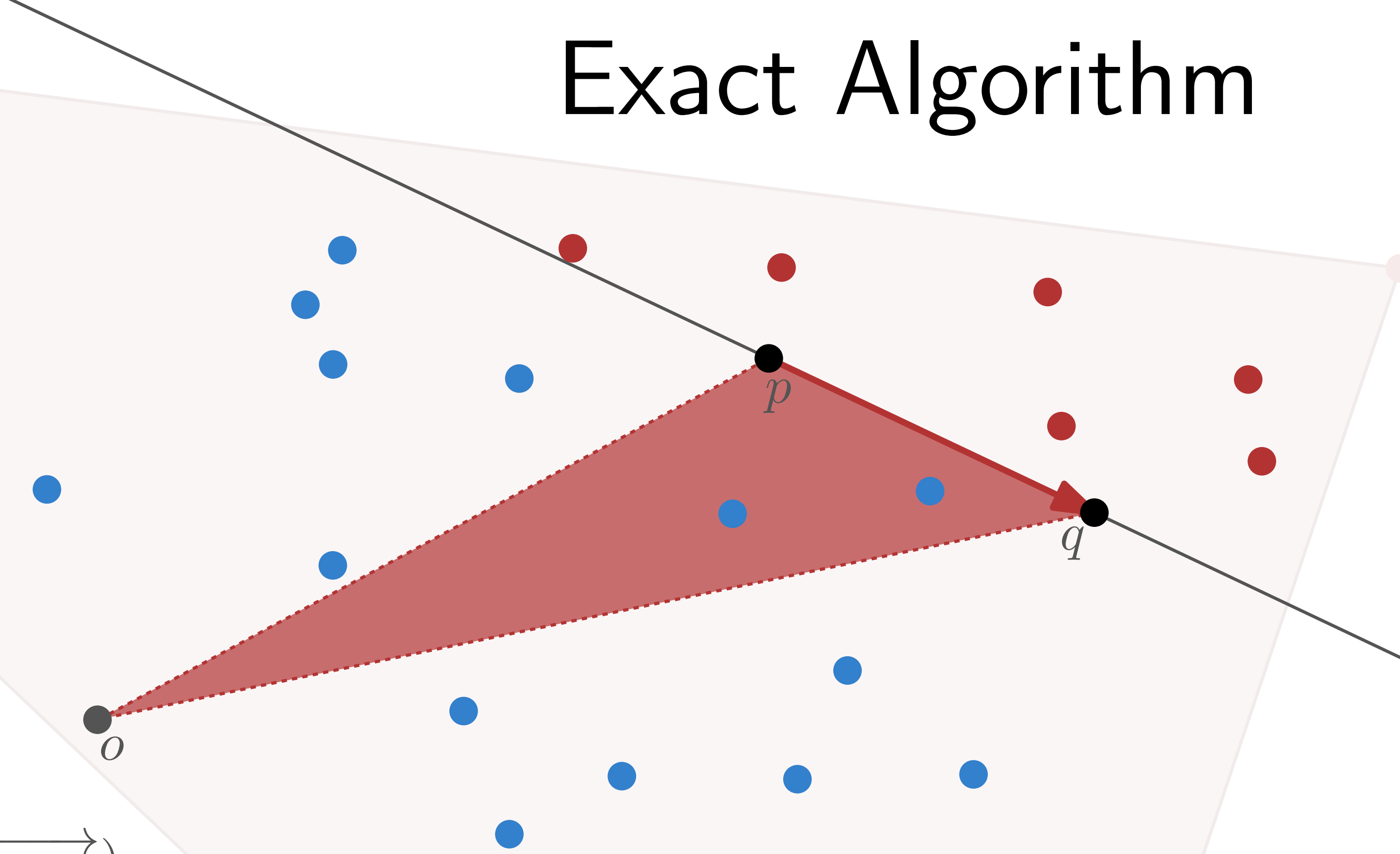

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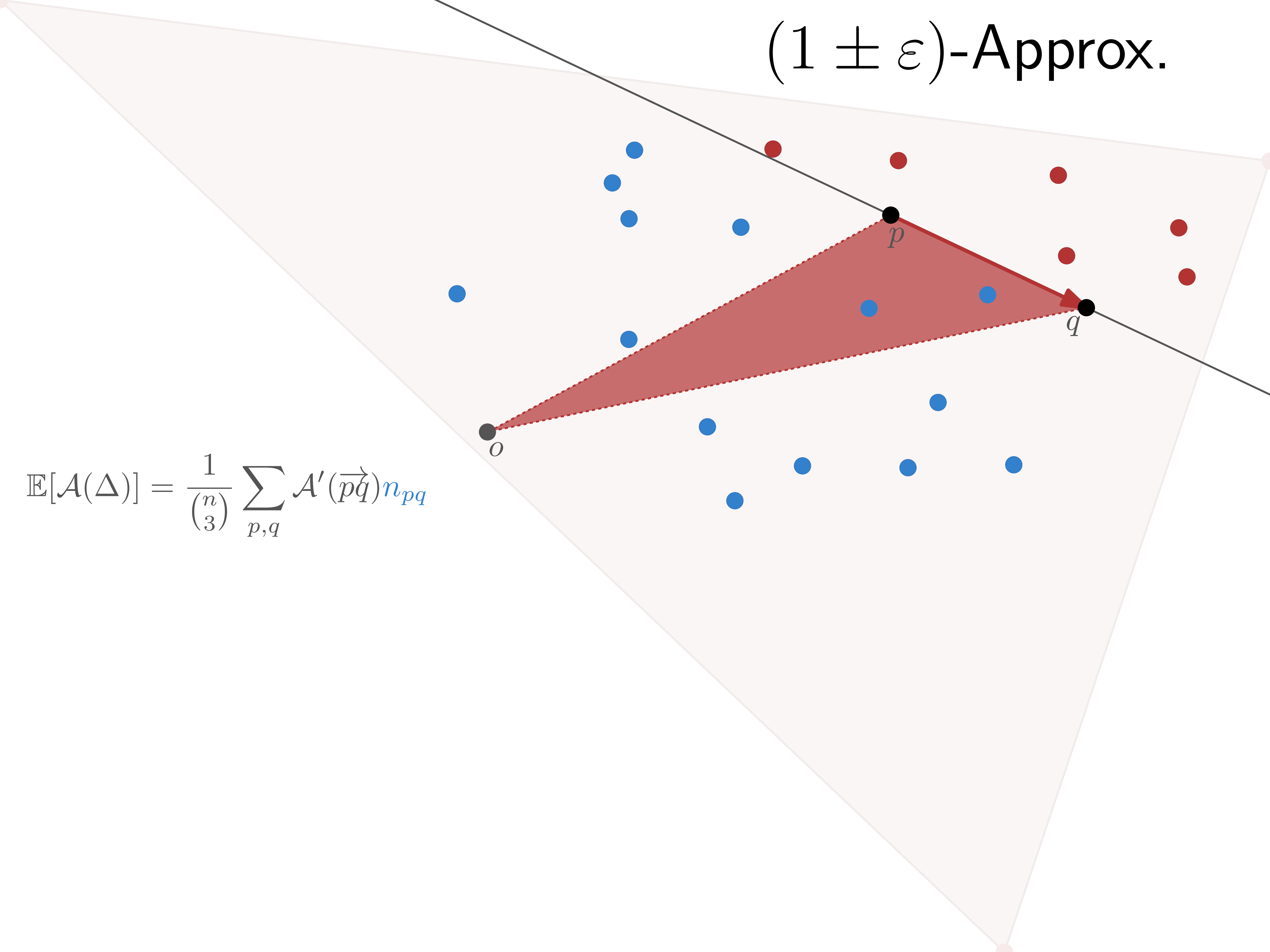
Theorem.
We can compute $\mathbb{E}[\mathcal{A}(\Delta)]$ in $O(n^2 \log n)$ time.

$(1 \pm \varepsilon)$-Approx.



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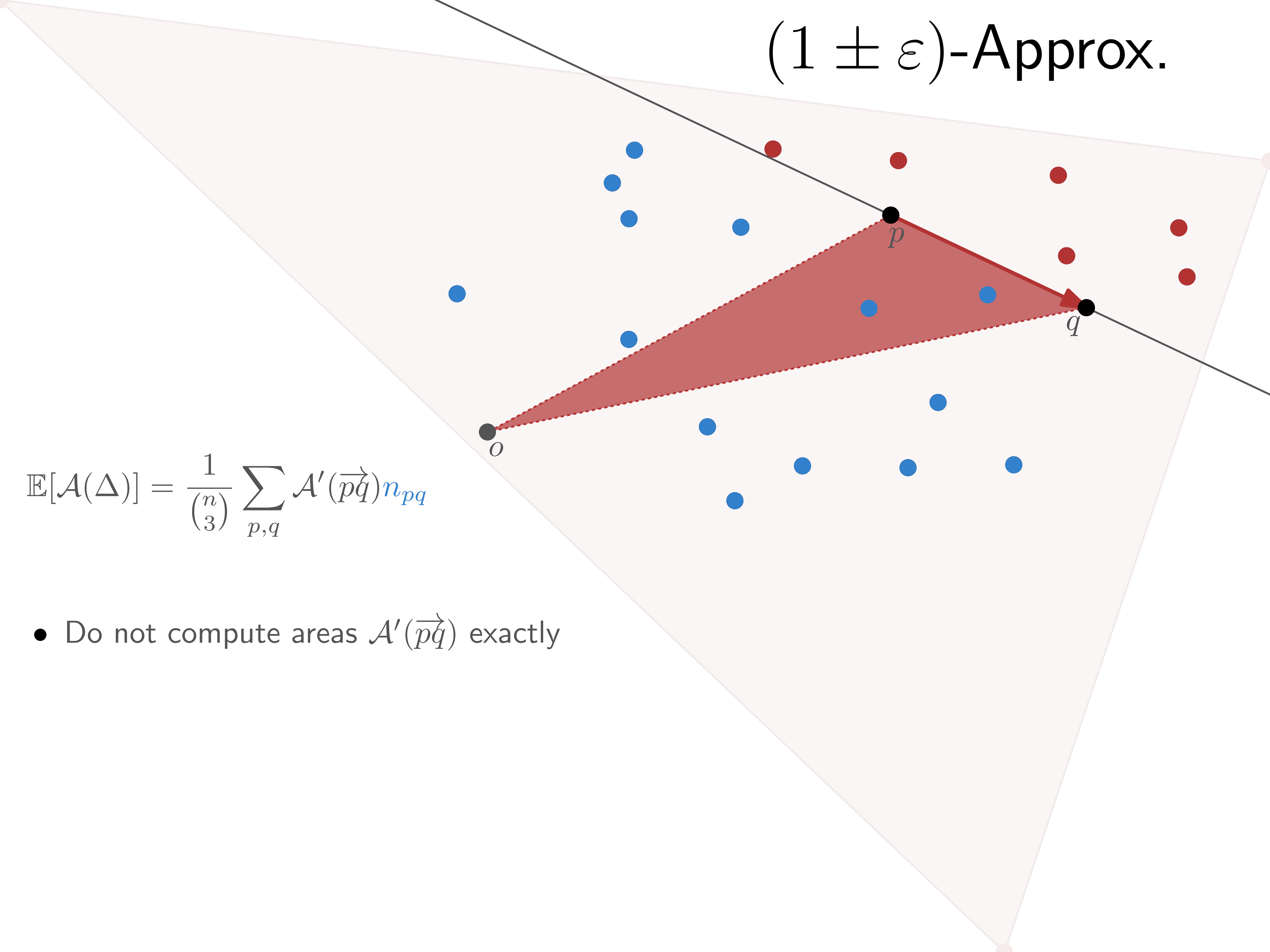

$$\mathbb{E}[\mathcal{A}(\Delta)] = \frac{1}{\binom{n}{3}} \sum_{p,q} \mathcal{A}'(\overrightarrow{pq}) n_{pq}$$

- Do not compute areas $\mathcal{A}'(\overrightarrow{pq})$ exactly

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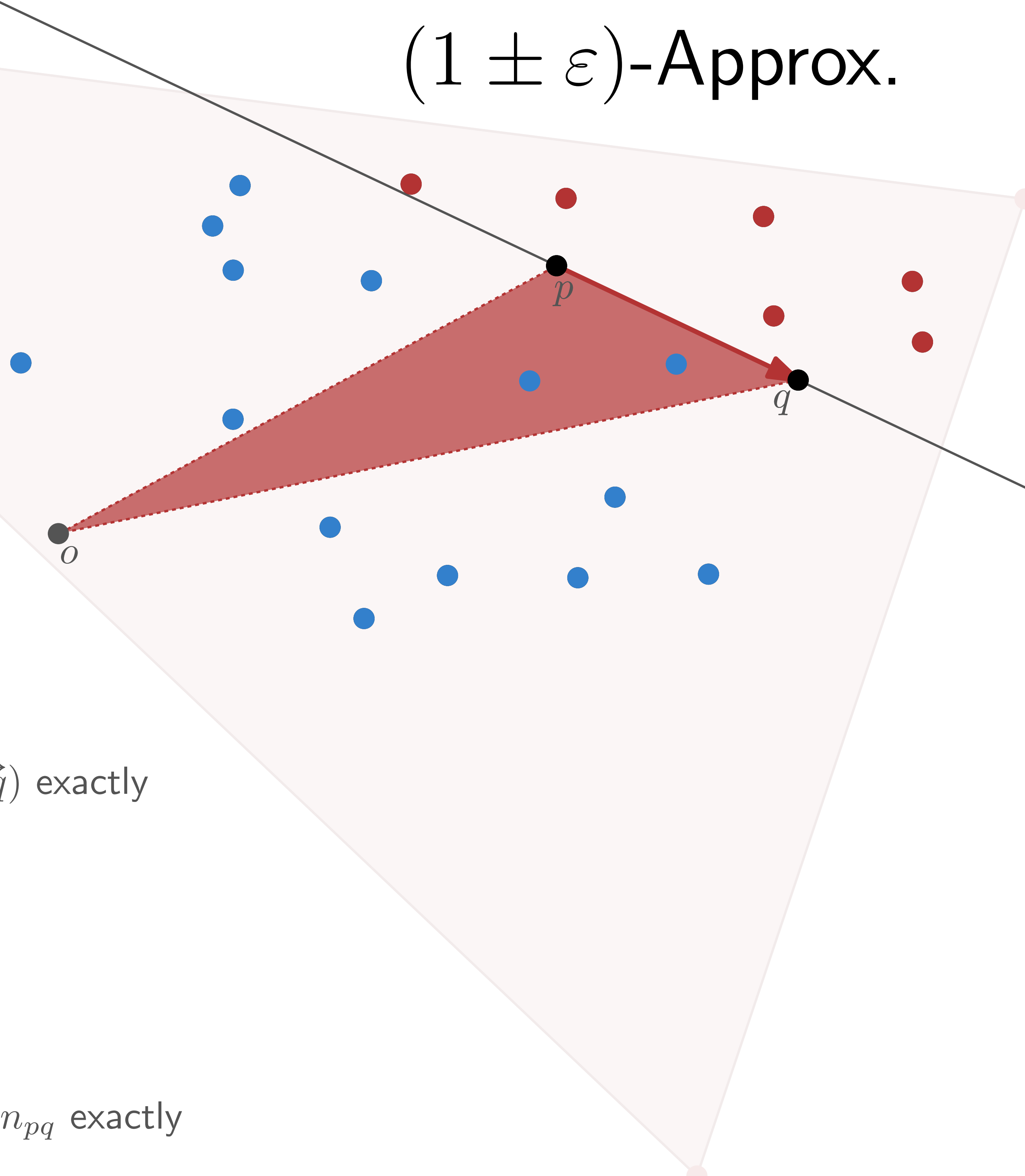

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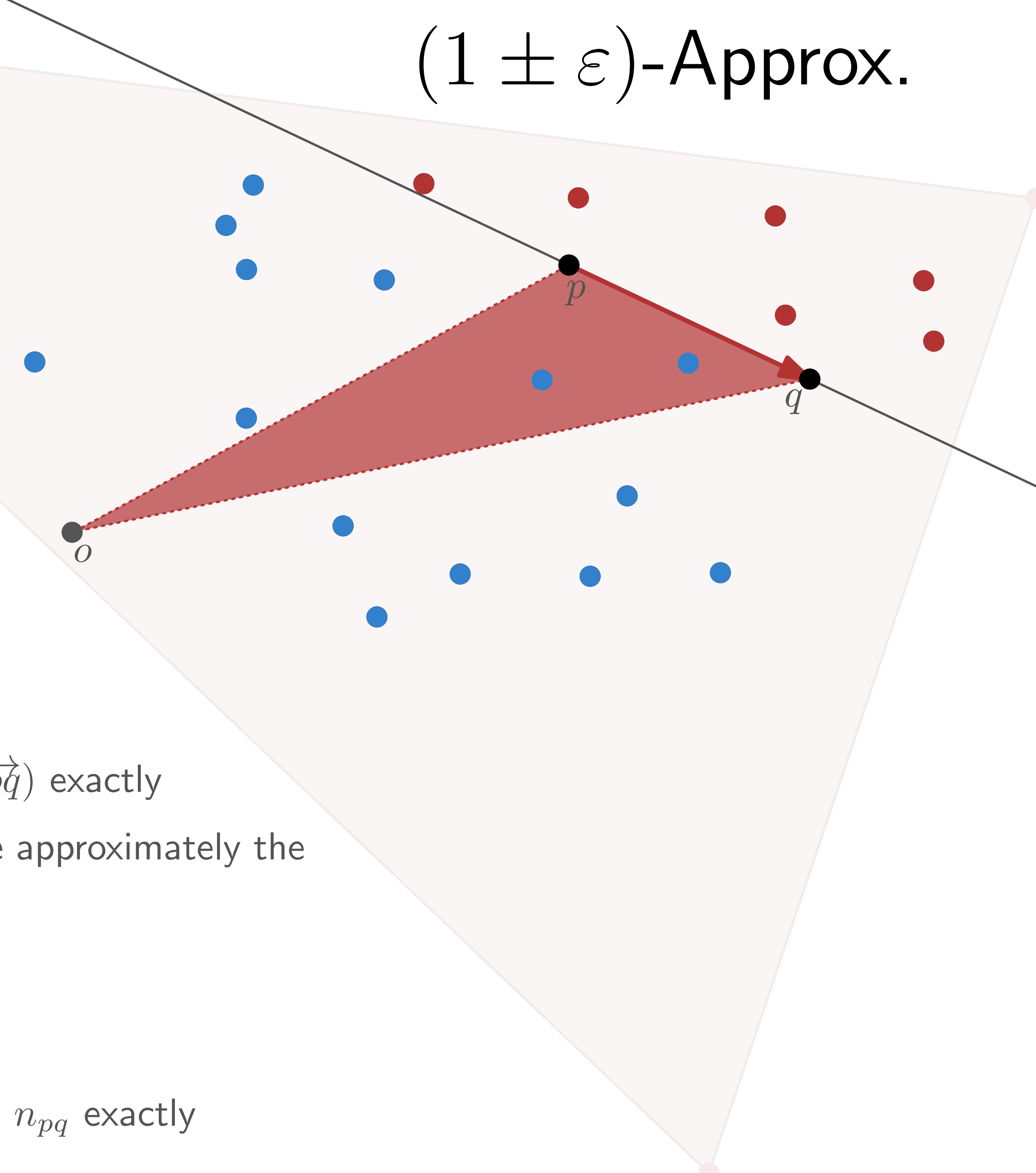

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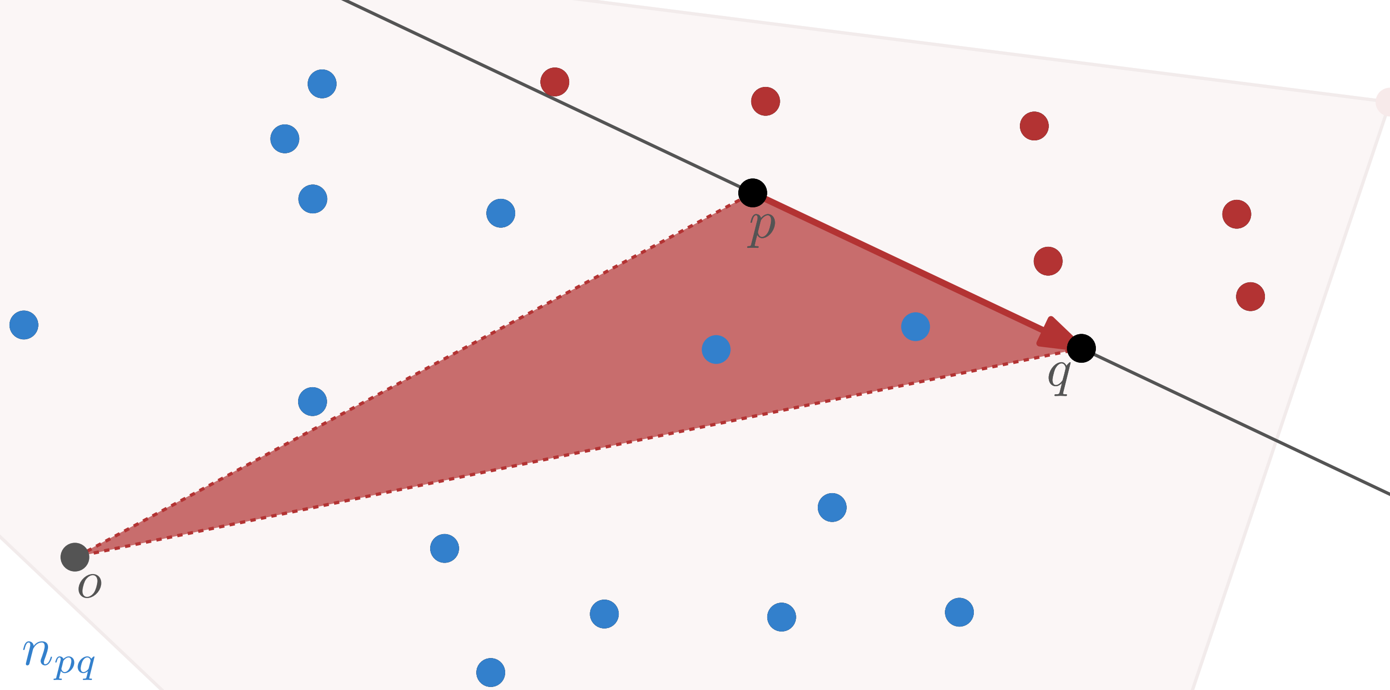

$$\mathbb{E}[\mathcal{A}(\Delta)] \approx \frac{1}{\binom{n}{3}} \sum_{G} \mathcal{A}'(G) \sum_{p,q \in G} n_{pq}$$

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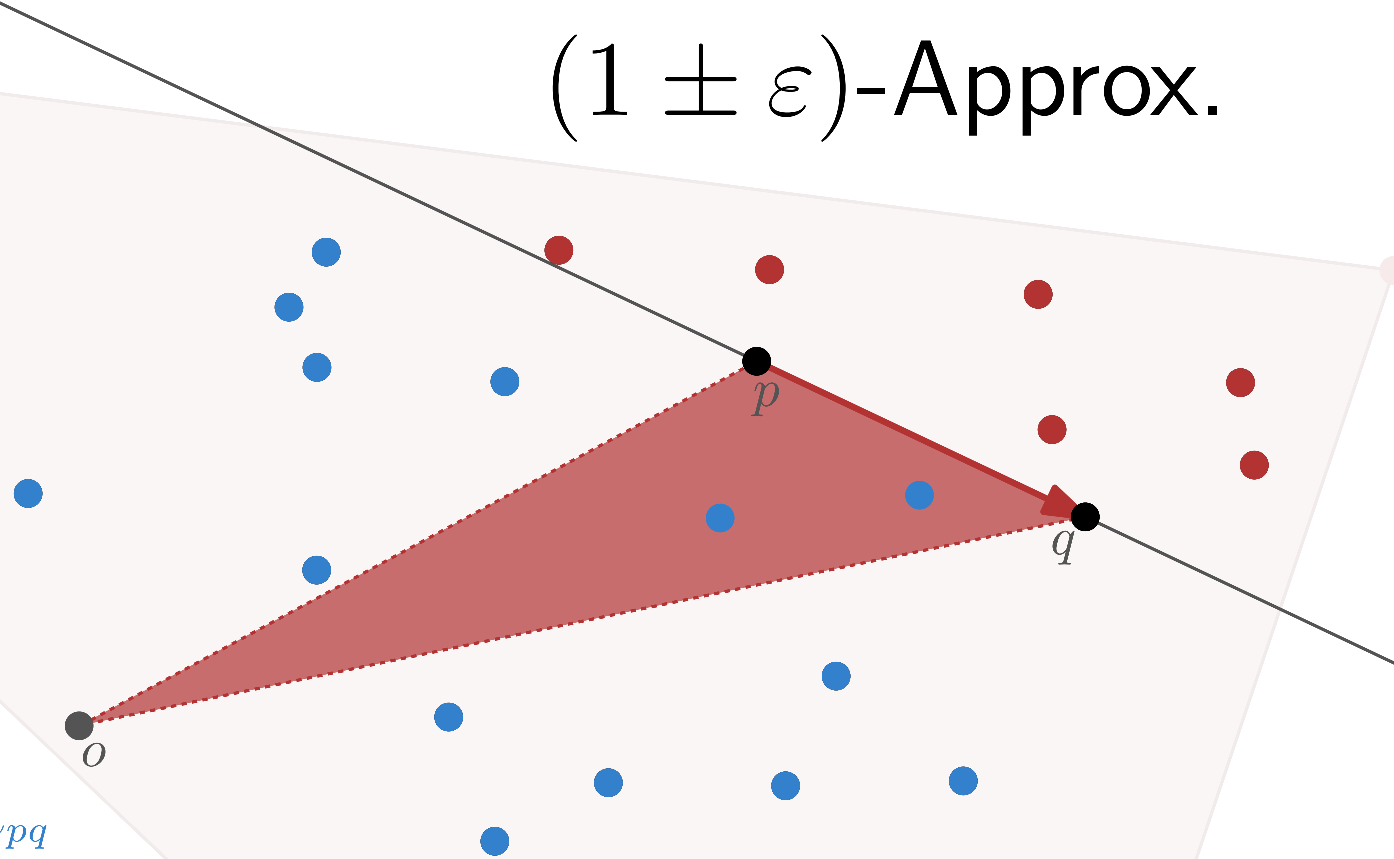

$$\mathbb{E}[\mathcal{A}(\Delta)] \approx \frac{1}{\binom{n}{3}} \sum_{(p,Q)} A_{pQ} \sum_{q \in Q} n_{pq}$$

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  Group pairs p, q that have approximately the same $\mathcal{A}'(\overrightarrow{pq})$

  $\Longrightarrow$ use a WSPD: $O(\frac{\rho^4}{\varepsilon^2} n \log n)$ pairs (p, Q) s.t. for all p and $q \in Q$ we have $(1-\varepsilon)\mathcal{A}'(\overrightarrow{pq}) \leq A_{pQ} \leq \mathcal{A}'(\overrightarrow{pq})$

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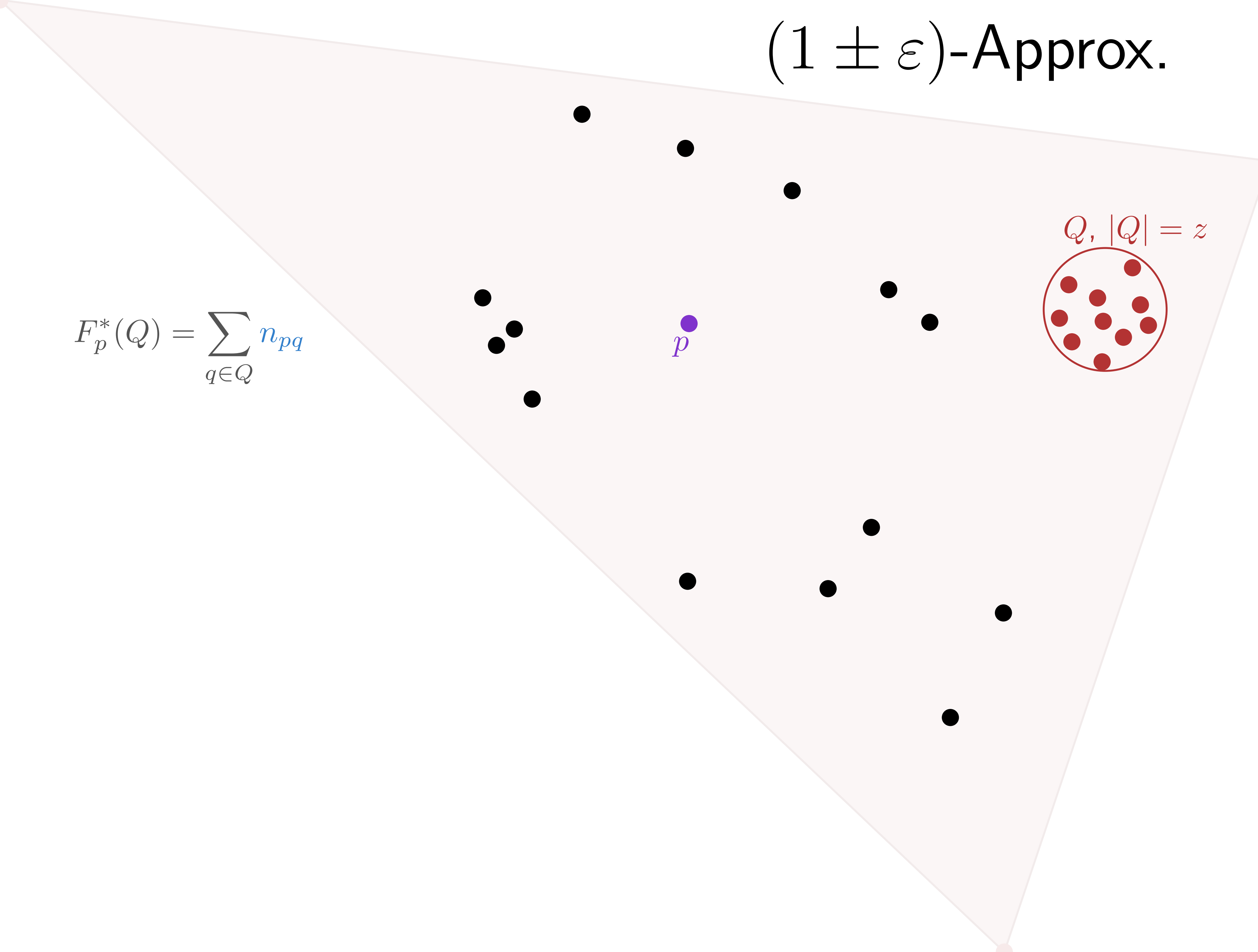

$$F_p^*(Q) = \sum_{q \in Q} n_{pq}$$

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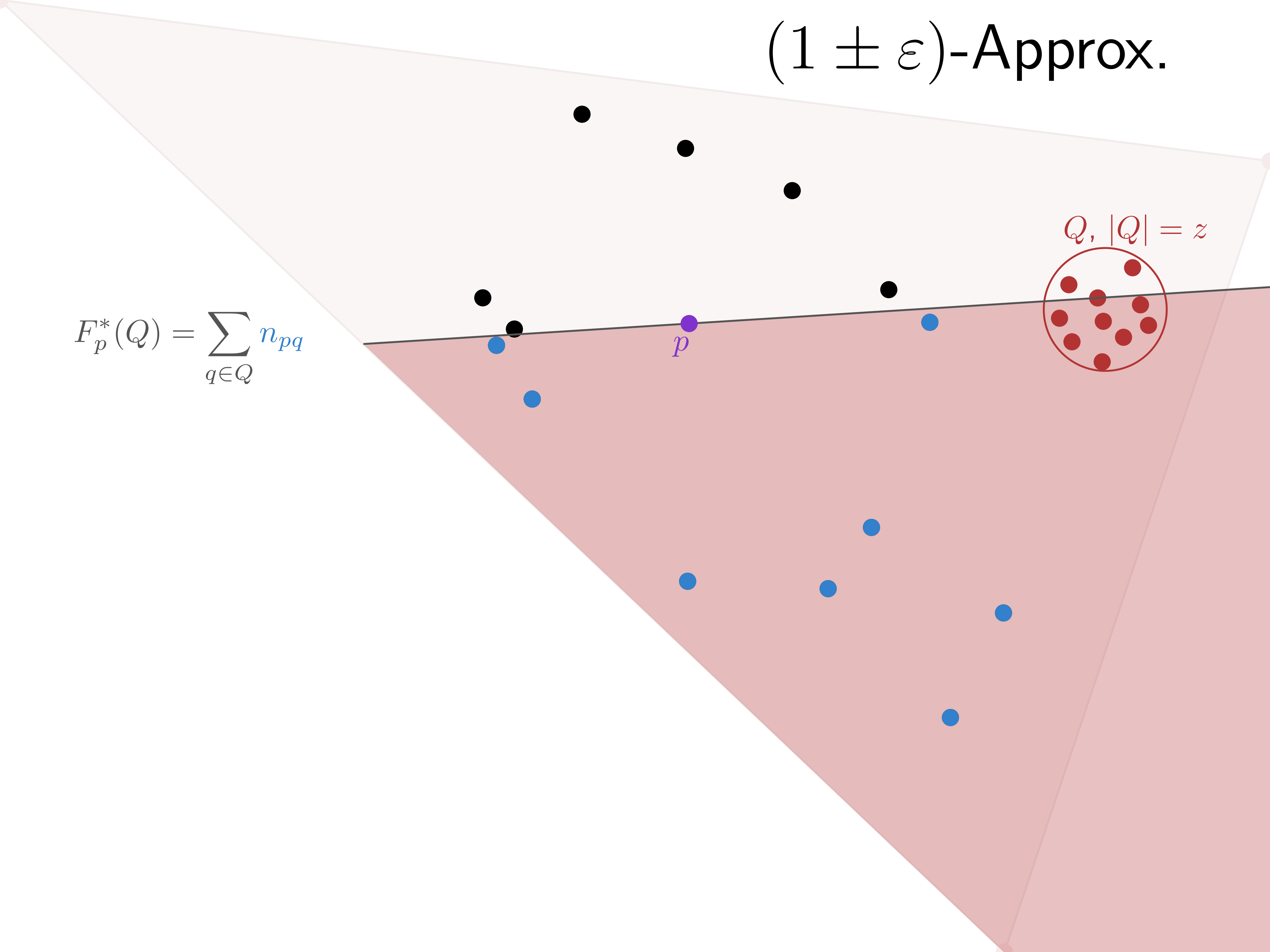

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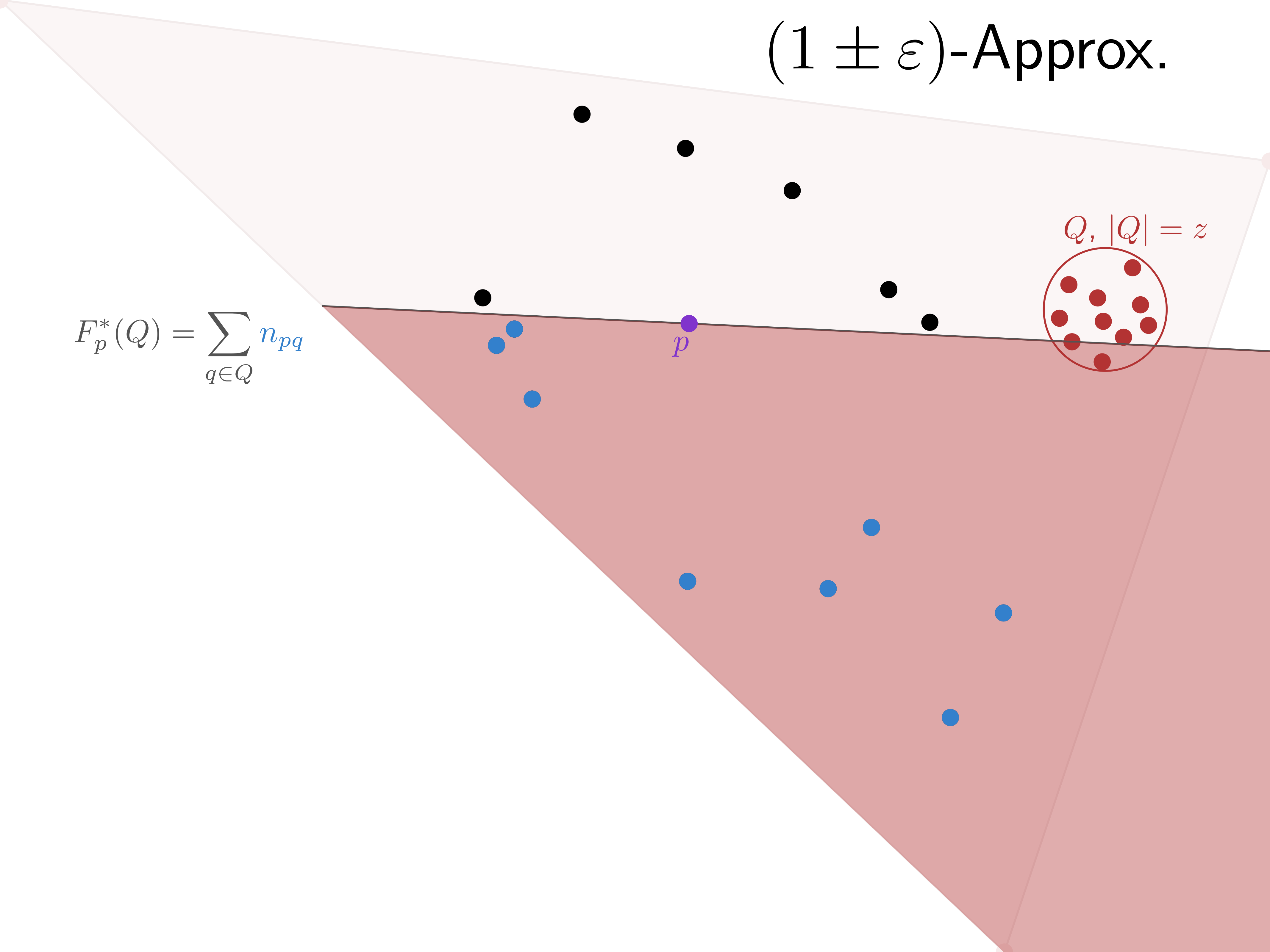

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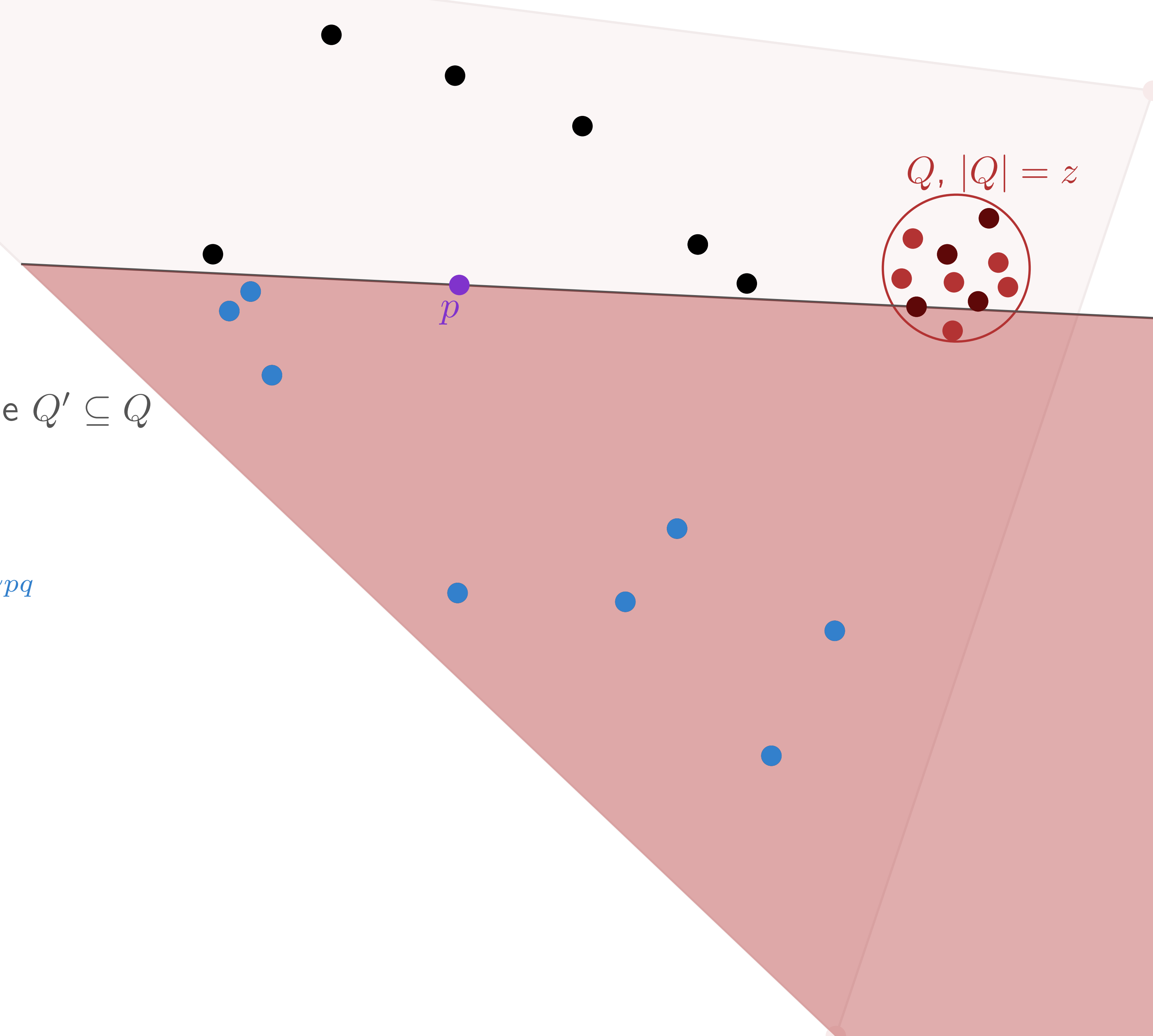
$$F_p^*(Q) = \sum_{q \in Q} n_{pq}$$

Take a random sample $Q' \subseteq Q$

$$F(Q) = \frac{z}{|Q'|} \sum_{q \in Q'} n_{pq}$$



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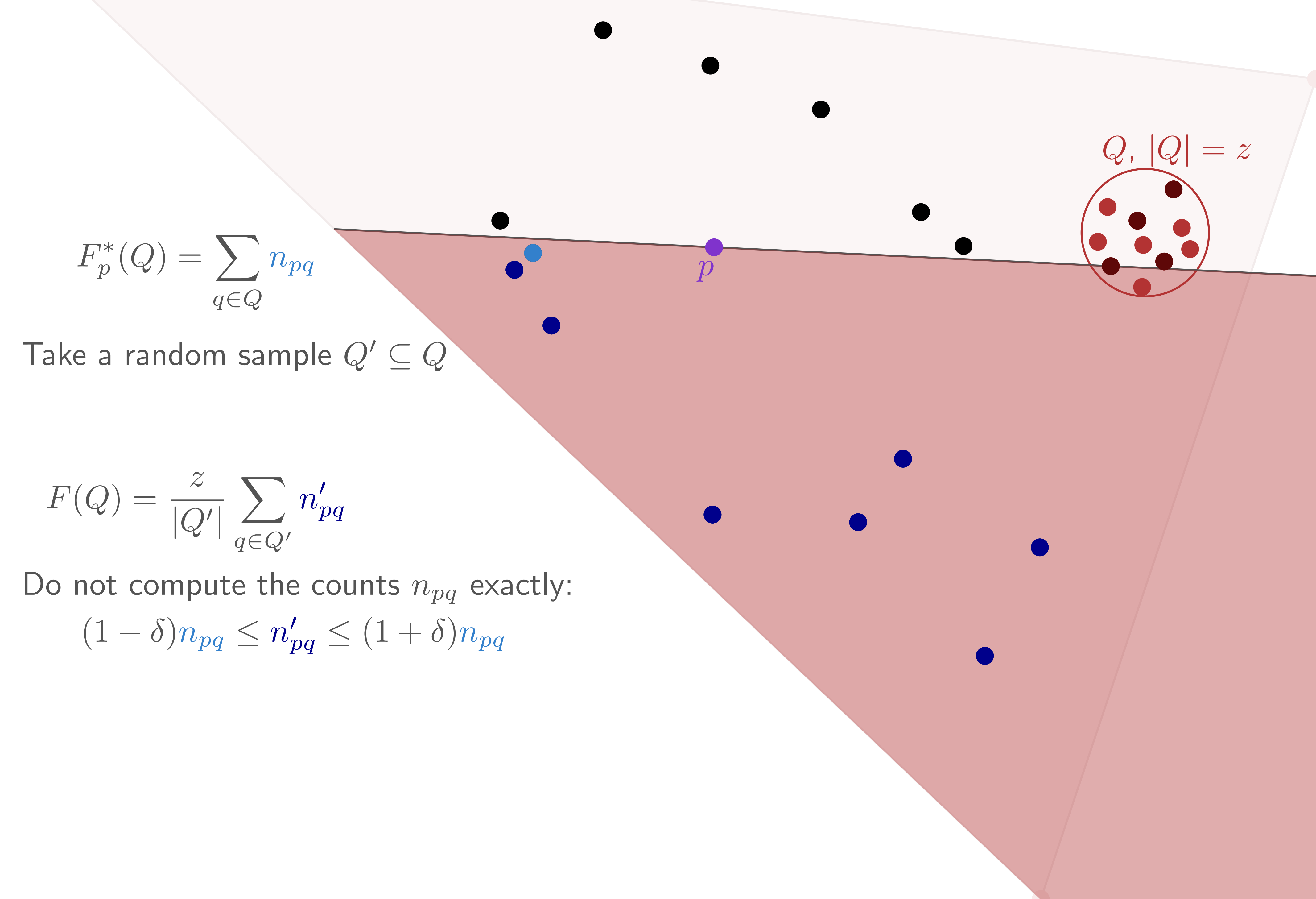

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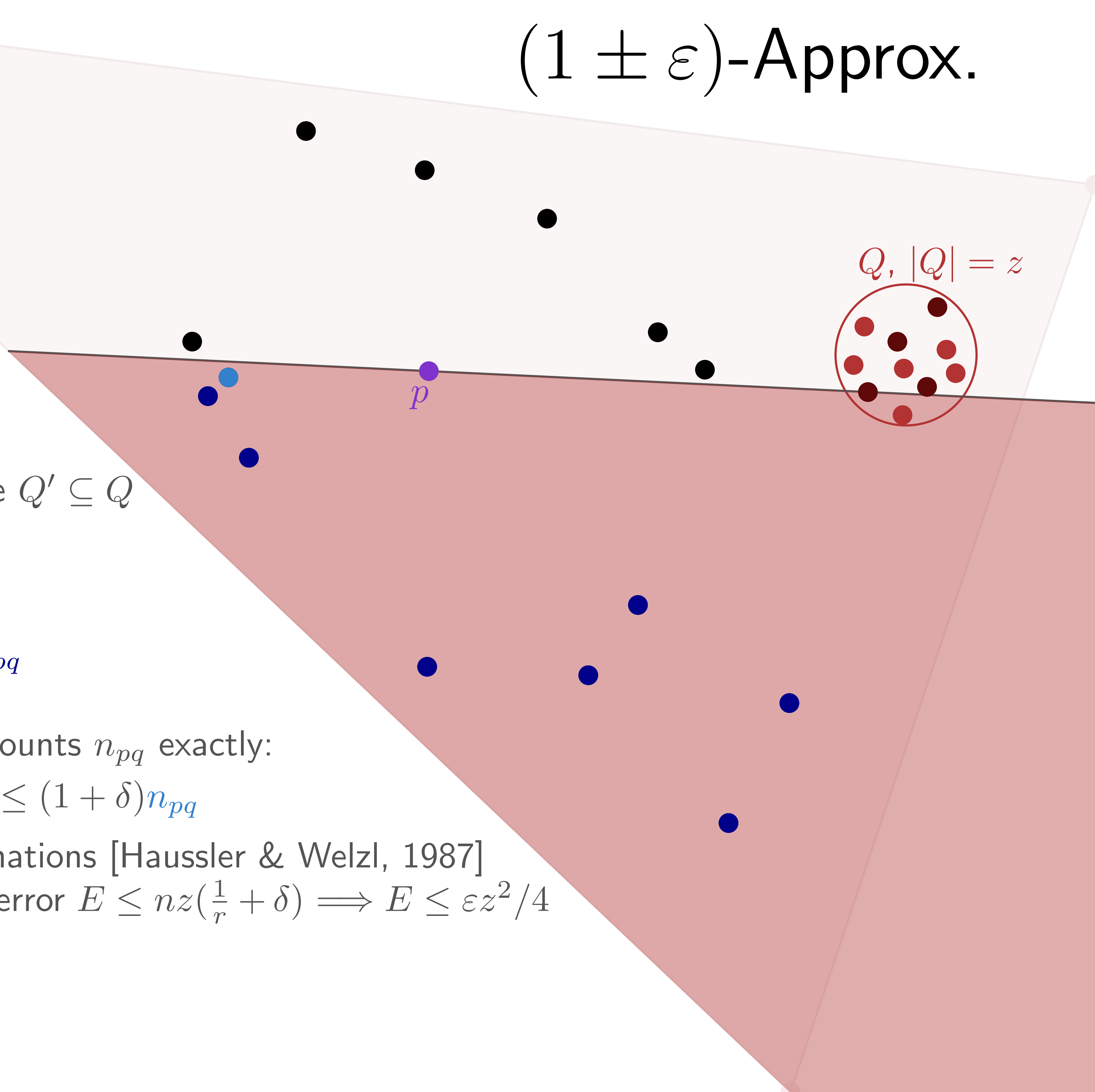

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$\Longrightarrow$ ε-nets/ε-approximations [Haussler & Welzl, 1987]

$\Longrightarrow$ .... $\Longrightarrow$ absolute error $E \leq nz(\frac{1}{r} + \delta) \Longrightarrow E \leq \varepsilon z^2/4$

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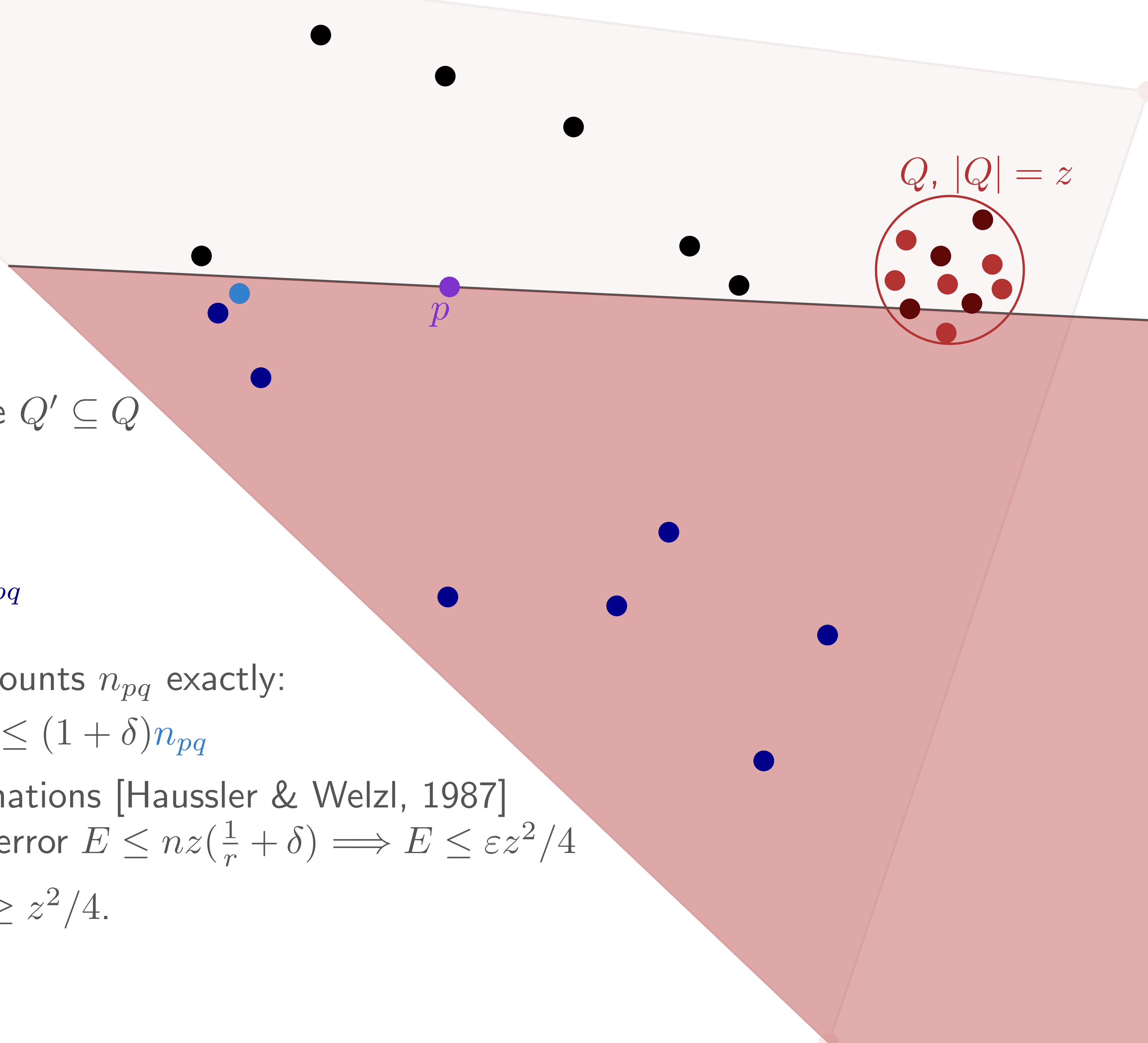

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$F_p^*(Q) \geq z(z-1)/2 \geq z^2/4.$

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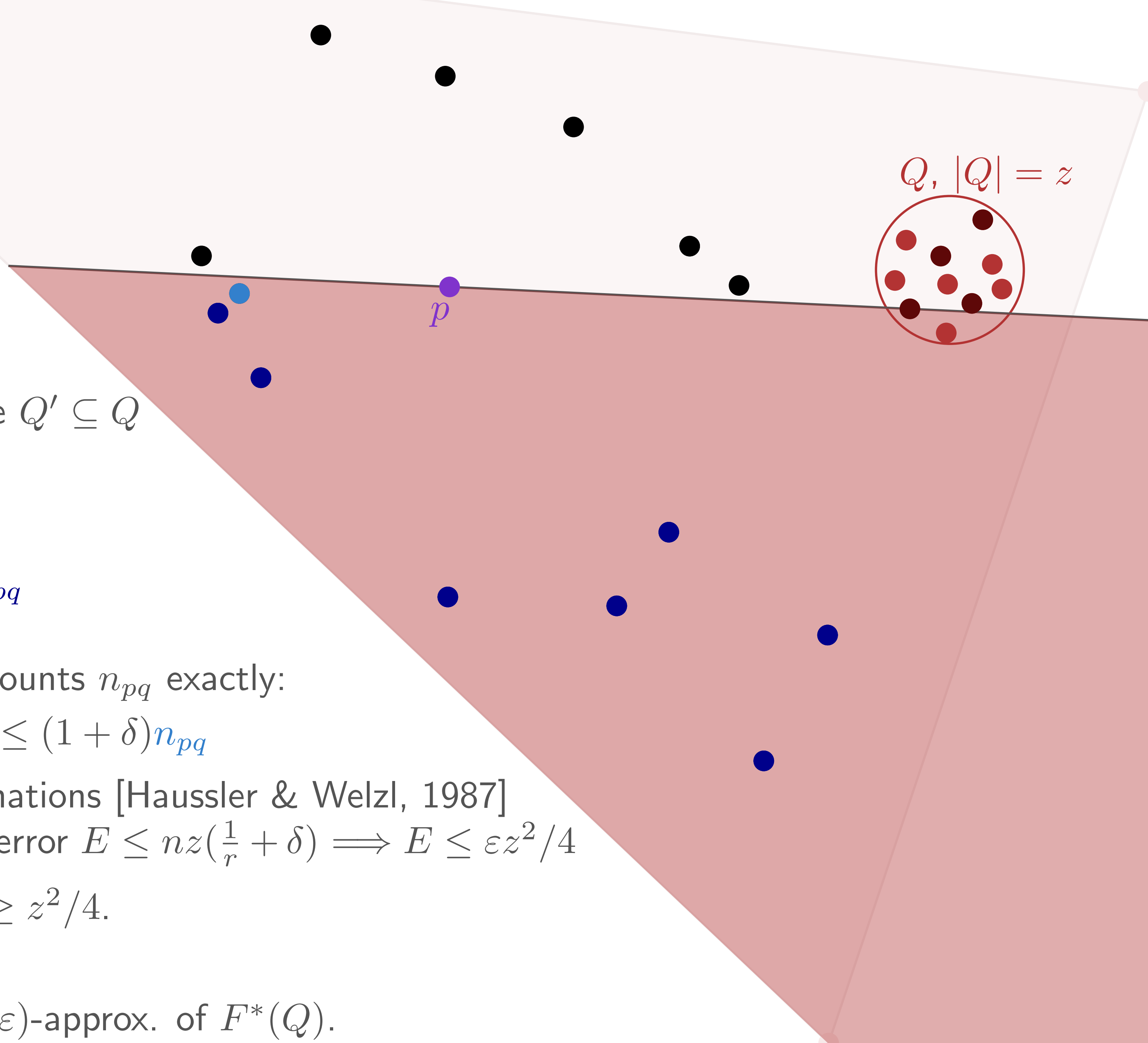

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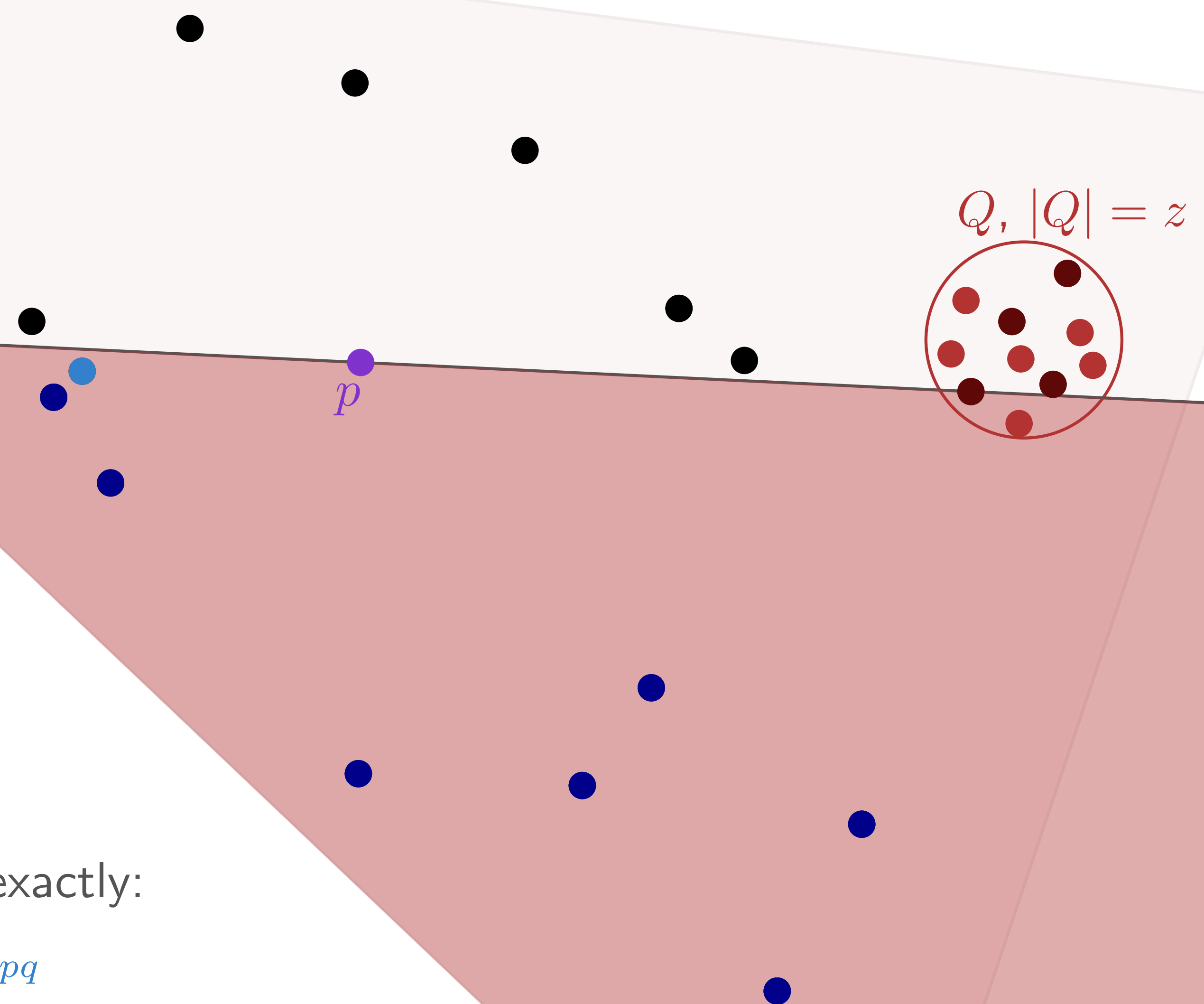
$F_p^*(Q) \geq z(z-1)/2 \geq z^2/4$.

Lemma.

Whp. $F(Q)$ is a $(1 \pm \varepsilon)$-approx. of $F^*(Q)$.

$(1 \pm \varepsilon)$-Approx.



$$F_p^*(Q) = \sum_{q \in Q} n_{pq}$$

Take a random sample $Q' \subseteq Q$
of size $\approx O((\frac{n}{\varepsilon})^{2/3})$

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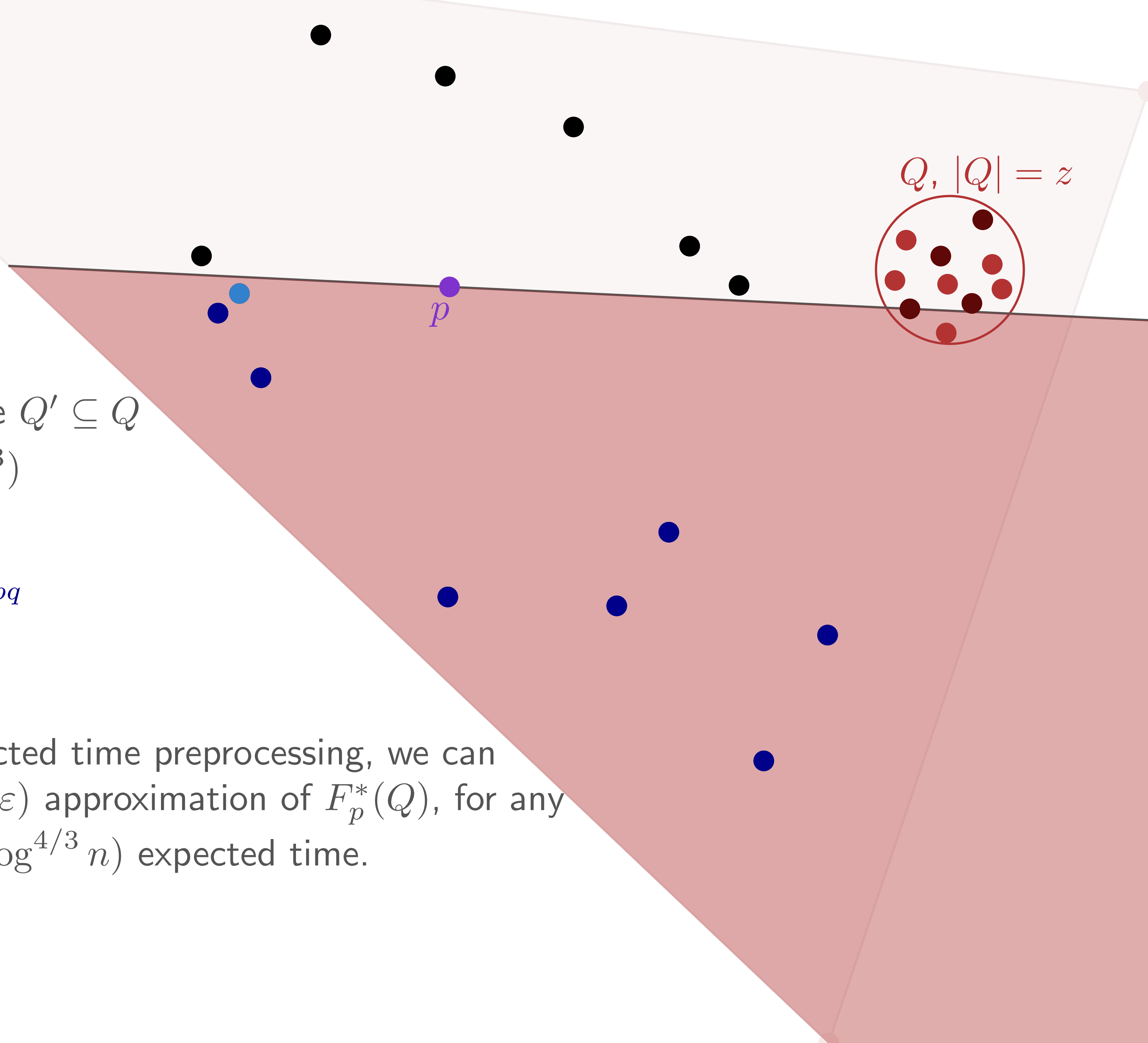

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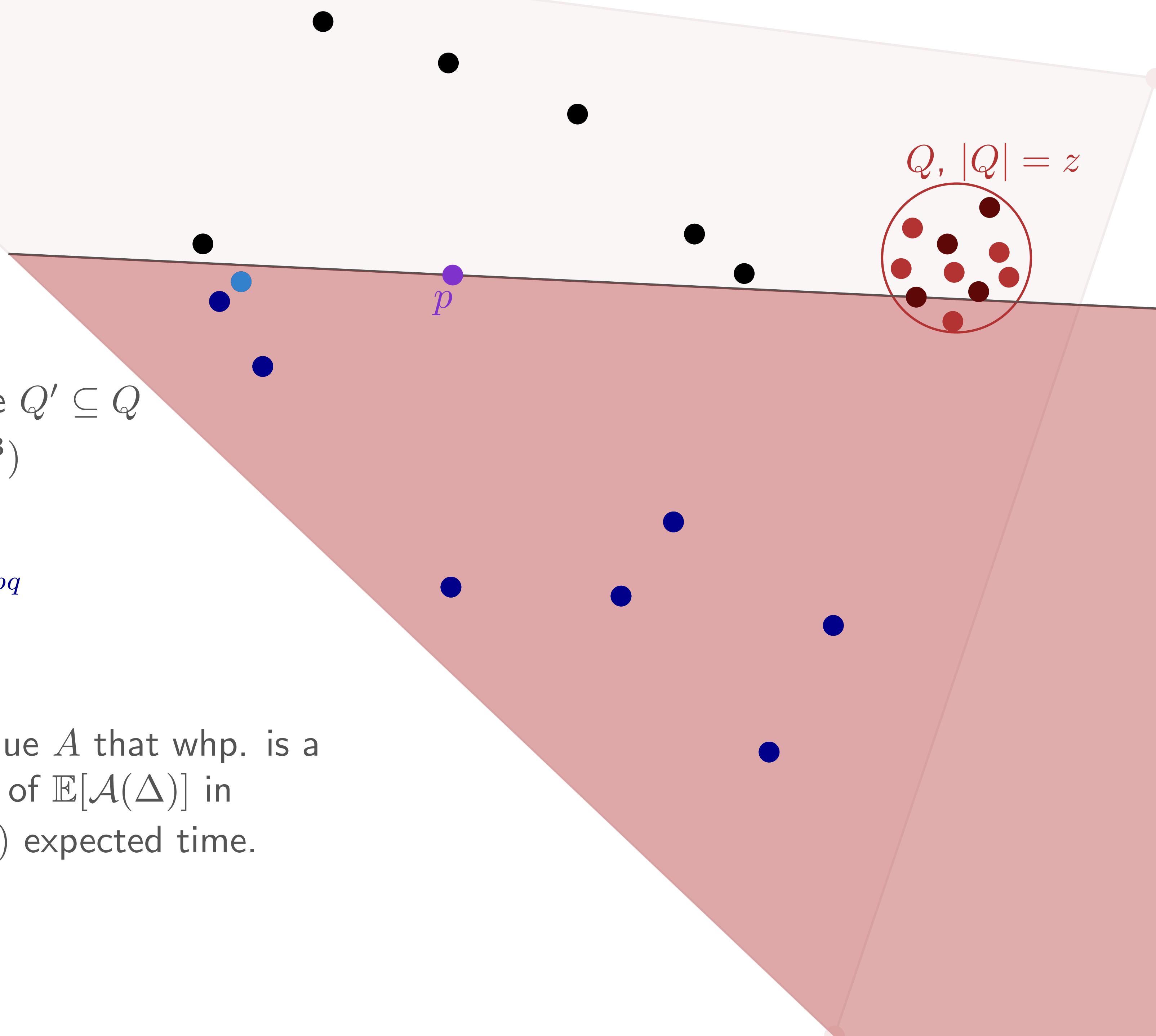
Lemma.
After $O(n \log n)$ expected time preprocessing, we can whp. compute a $(1 \pm \varepsilon)$ approximation of $F_p^*(Q)$, for any (p, Q) in $O((n/\varepsilon)^{2/3} \log^{4/3} n)$ expected time.

$(1 \pm \varepsilon)$-Approx.



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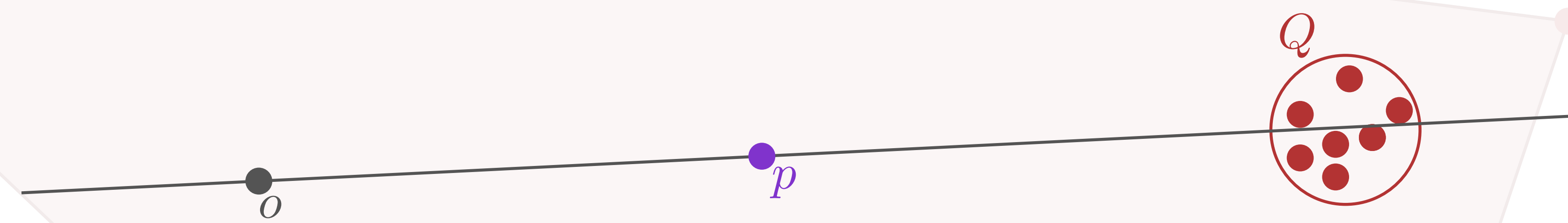
Theorem.
We can compute a value A that whp. is a $(1 \pm \varepsilon)$-approximation of $\mathbb{E}[\mathcal{A}(\Delta)]$ in $O(\frac{1}{\varepsilon^{8/3}} \rho^4 n^{5/3} \log^{7/3} n)$ expected time.

Future Work



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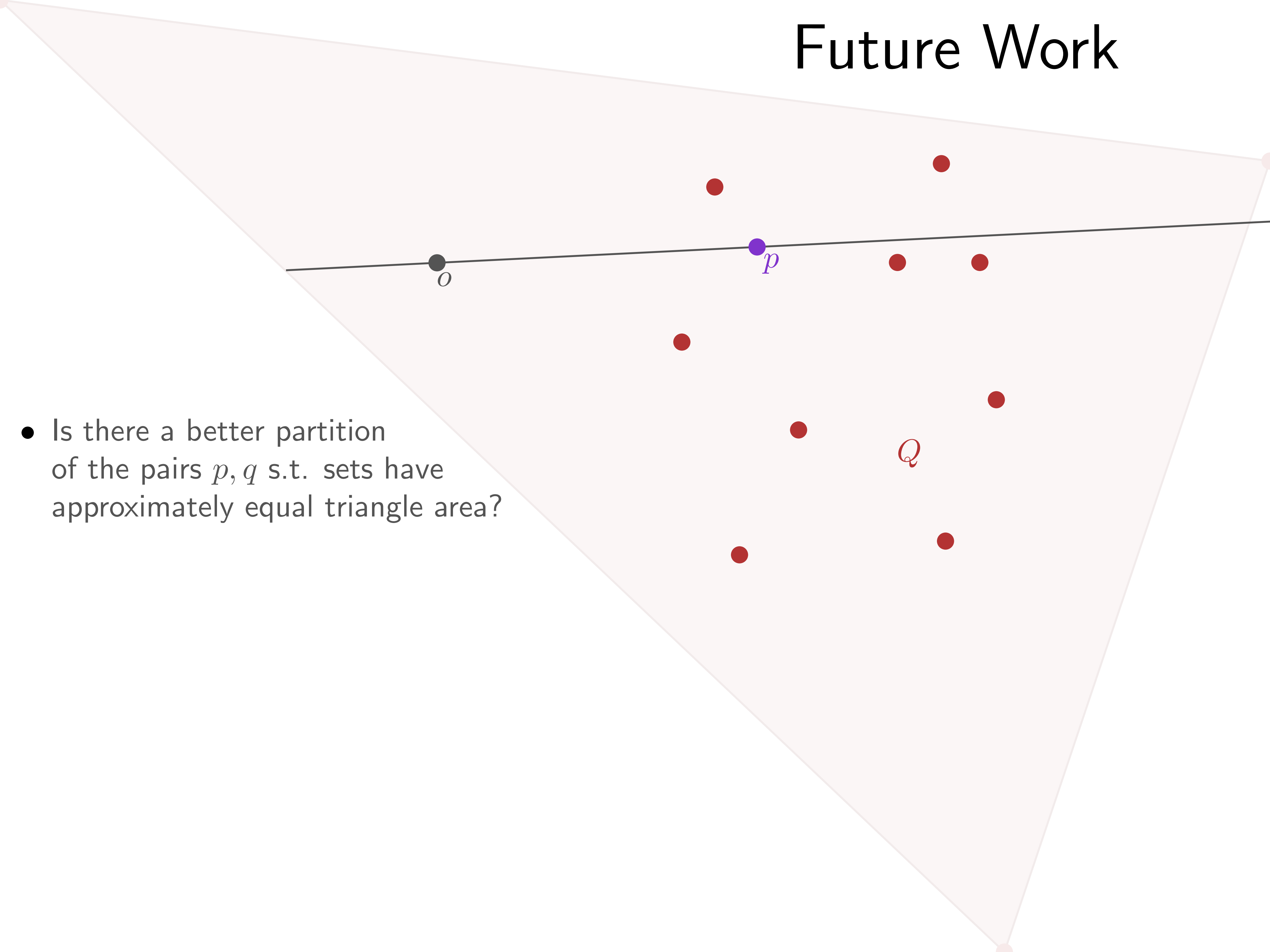

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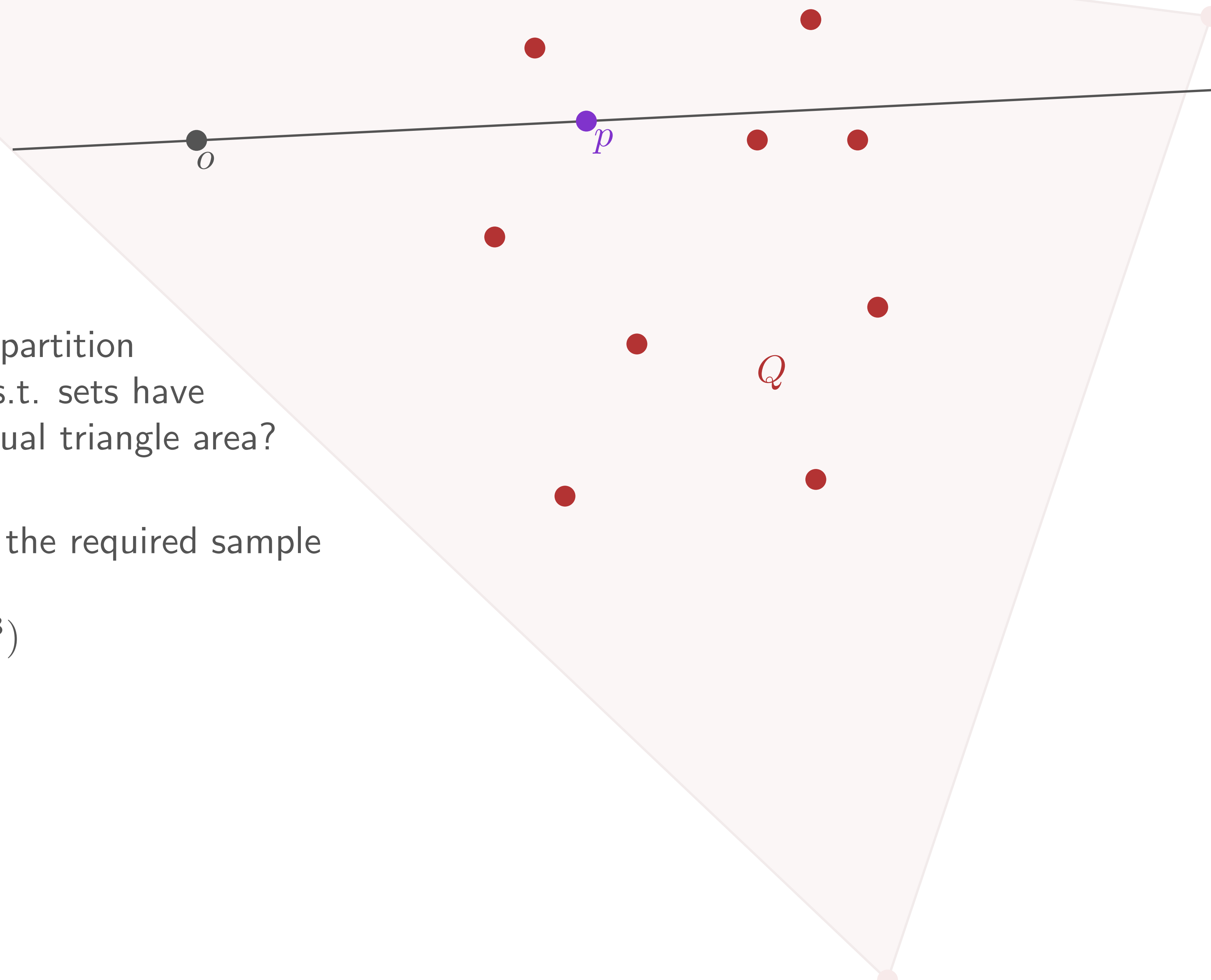

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- Can we decrease the required sample size?
  $|Q'| \approx O((\frac{n}{\varepsilon})^{2/3})$

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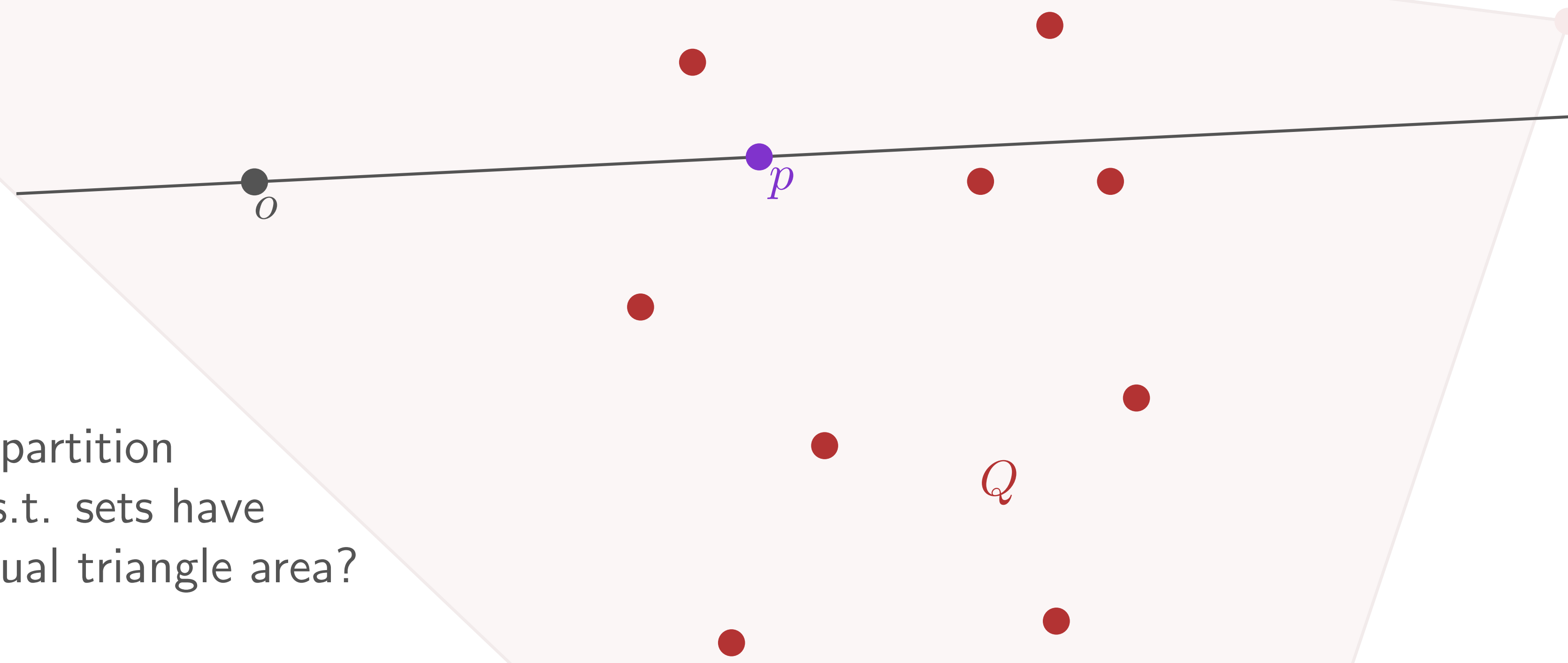

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  $|Q'| \approx O((\frac{n}{\varepsilon})^{2/3})$

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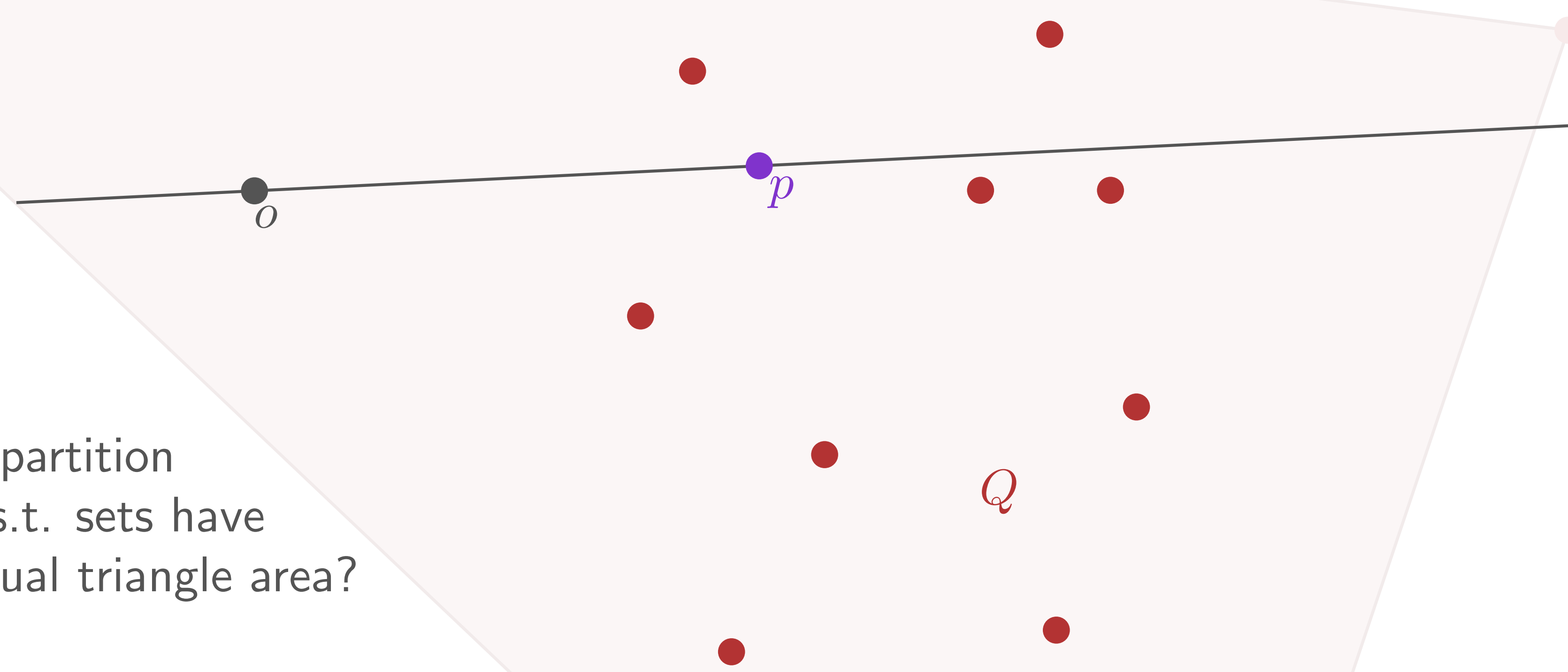
Future Work



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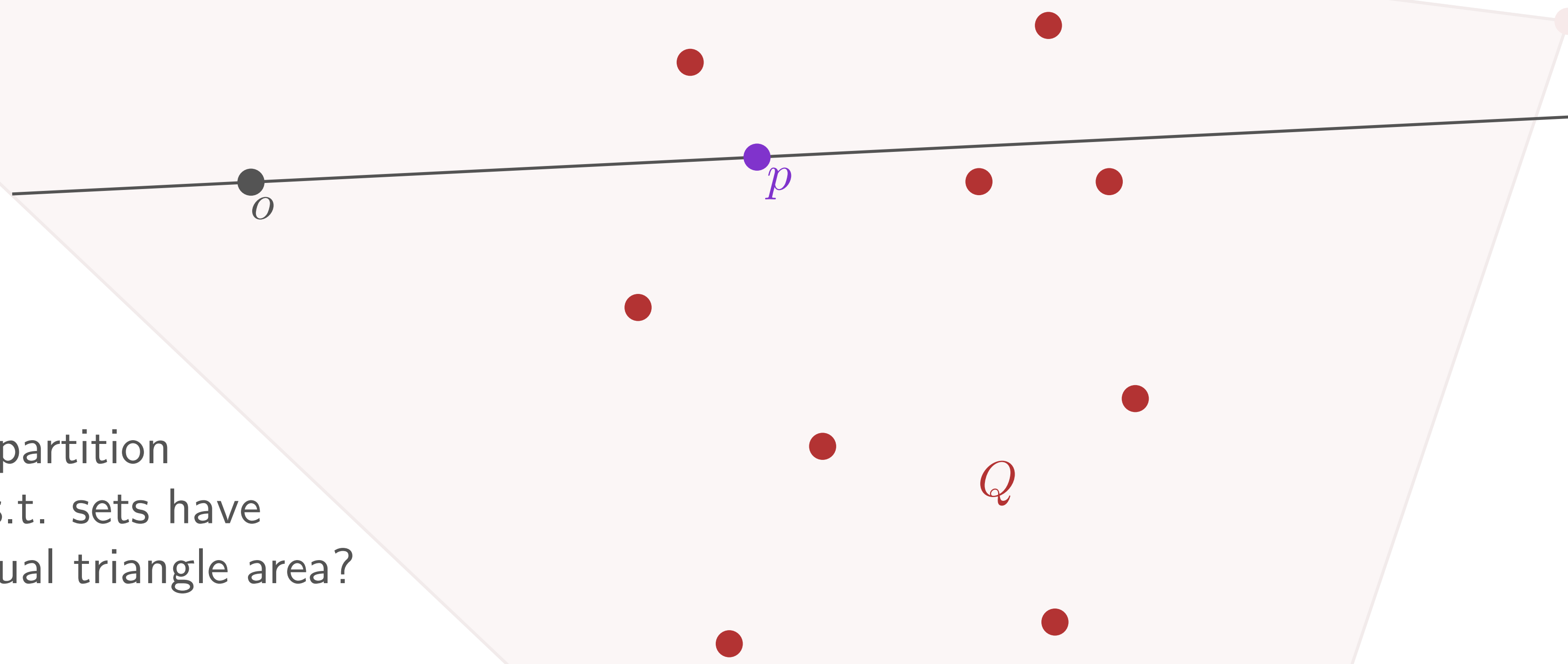
Future Work



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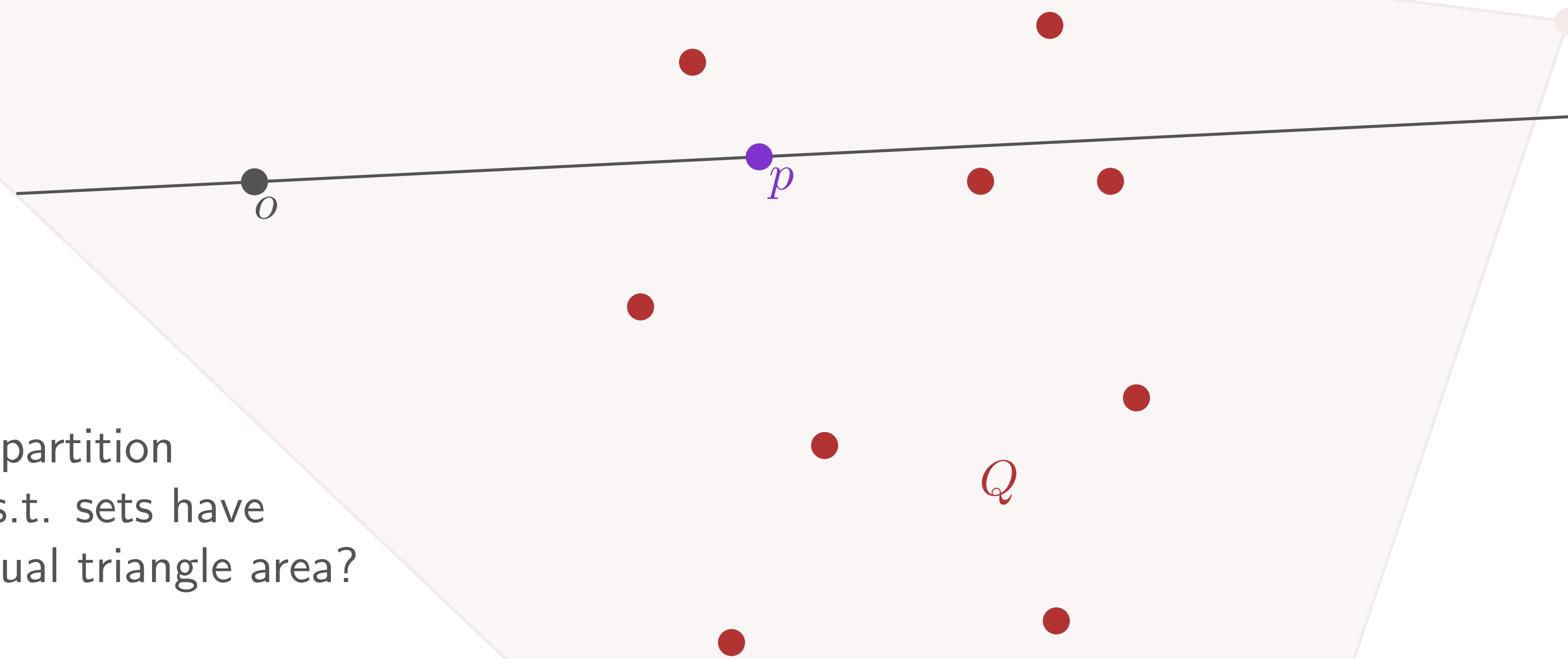
Future Work



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Thank you!